The Central Limit Theorem problem for t-normed sums of random upper semicontinuous functions

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1 Introduction

We will present the problem of the Central Limit Theorem for random upper semicontinuous functions. Let us begin by providing the context for that problem. We need to present the space in which we will work, and specify the notions of random element of that space, convergence and the operations used to average random elements. Then we will make some historical remarks on limit theorems in this setting.

1. Space of u.s.c. functions. Amongst upper semicontinuous (u.s.c.) functions we refer, more precisely, to the space $F_c$ of all functions $u$ from a separable Banach space $E$ to $[0,1]$ such that
   (a) Each level set $u_\alpha = \{x \mid u(x) \geq \alpha\}$ is closed, convex and non-empty, for $\alpha \in (0, 1]$,
   (b) The support of $u$ is relatively compact (we will denote its closure by $u_0$).

2. Random u.s.c. functions. A random u.s.c. function is a mapping $X$ from a measurable space $(\Omega, \mathcal{A})$ to $F_c$ such that for each level mapping $X_\alpha$,
   \[ \{\omega \in \Omega \mid X_\alpha(\omega) \cap G \neq \emptyset\} \in \mathcal{A} \]
   for every open set $G \subset E$. See [10, 3].

3. Convergence. Convergence in $F_c$ is meant in one of the following senses:
(a) In the metric \( d_\infty(u, v) = \sup_{\alpha \in [0, 1]} d_H(u_\alpha, v_\alpha) \), where
\[
d_H(A, C) = \max\{\sup_{x \in A} \inf_{y \in C} ||x - y||, \sup_{x \in C} \inf_{y \in A} ||x - y||\}
\]
is the Hausdorff metric.

(b) In the metric \( d_p(u, v) = (\int_0^1 d_H(u_\alpha, v_\alpha)^p d\alpha)^{1/p} \), for some \( p \in [1, \infty) \).

(c) In the weak topology \( \tau \) generated by the level mappings \( u \mapsto u_\alpha \), namely \( u_n \to u \) if \( d_H((u_n)_\alpha, u_\alpha) \to 0 \) for each \( \alpha \in (0, 1] \).

4. Algebraic structure. There remains to specify the operations used in \( \mathcal{F}_c \). Maybe surprisingly for the unaware reader, these are not the usual pointwise operations.

Let \( \triangledown \) be a triangular norm (usually called just \( t \)-norm), namely a binary operation \( \triangledown : [0, 1] \times [0, 1] \to [0, 1] \) which is associative, non-decreasing in each argument and has 1 as neutral element (it follows that \( \triangledown \) is in fact an Abelian monoid and 0 is an annihilator) [5]. We additionally assume that \( \triangledown \) is continuous.

Denoting by \( I_A \) the indicator function of a set \( A \), we define
\[
(u + v)(x) = \sup_{x_1 + x_2 = x} u(x_1) \triangledown v(x_2),
\]
\[
(\lambda u)(x) = u(\lambda^{-1} x), \lambda \neq 0, \quad \text{and} \quad (0u)(x) = I_{\{0\}}(x).
\]
The meaning of these operations is not immediately apparent, but notice that
\[
I_A + I_C = I_{A+C}, \quad \lambda I_A = I_{\lambda A},
\]
so that these operations generalize the elementwise operations on sets. Notice also that convergence of indicator functions in any of the senses in (3) is the same as convergence of sets in the Hausdorff metric \( d_H \).

Hence, we have defined a structure on \( \mathcal{F}_c \) which is an extension of the algebraic and metric structure of the space of non-empty compact convex subsets of \( E \). An element of \( \mathcal{F}_c \) is thus a generalized indicator function allowed to take partial membership values in \([0, 1]\). This is usually referred to as a fuzzy set [14].

Notice that addition is parameterized in that it depends on the chosen \( t \)-norm. In order to emphasize that fact, whenever \( X \) is a random u.s.c. function and \( \{X_n\}_n \) are i.i.d. as \( X \), we set \( S_n(X; \triangledown) = n^{-1}\sum_{i=1}^n X_i \), where of course the addition in the right-hand side is the one defined via the \( t \)-norm \( \triangledown \).
5. Historical remarks. Limit theorems for random u.s.c. functions have, until very recently, been studied only for the choice $\top = \min$, although t-normed constructions have appeared in books at least since 1980 [2]. For the choice $\top = \min$, it holds that $(u + v)_\alpha = u_\alpha + v_\alpha$, a very useful identity. Klement, Puri and Ralescu [6] proved the Strong Law of Large Numbers and the Central Limit Theorem in the metric $d_1$ when $E = \mathbb{R}^d$ and under a more stringent definition of measurability. See also e.g. [1, 8, 11] for the modern version of the SLLN. As regards the CLT, an unnatural Lipschitzianity condition in [6] was removed by [9], and recently Li et al. [7] provide the best version currently available with $d_\infty$ convergence.

Unfortunately, the CLT of Li et al. is stated under another quite unnatural assumption already inherited from the literature of random sets [4], namely that the random u.s.c. function $X$ is such that its support is always included in the cone generated by a fixed compact subset $K \subset E$. This technical assumption is vacuous in finite-dimensional spaces (since the unit ball is compact and generates the whole space) but unnaturally stringent in general.

As regards the problem involving a general t-norm, we have proven the SLLN for $E = \mathbb{R}^d$ in the senses of $\tau$ and $d_p$ [13]. A counterexample shows that the SLLN in $d_\infty$ fails when $\top$ is the product [12].

2 The Central Limit problems for t-normed sums

We have already mentioned that the best available version of the CLT for random u.s.c. functions\textsuperscript{1} (under the minimum t-norm) is not fully satisfactory, and that no CLT exists at all for t-norms other than the minimum.

Problem 1. Drop the technical condition in the CLT for random u.s.c. functions of Li et al. [7] and, as a by-product, in the CLT for random compact sets of Giné et al. [4].

This problem is important for t-normed sums because a good version of the CLT for random sets is needed in order to hope to derive good versions of the CLT for t-normed sums.

It is known [12] that the CLT does not hold in general for t-normed sums, however it holds for the minimum t-norm. Hence we have a somewhat vague

\textsuperscript{1}Or even for random sets.
Problem 2. Understand the interplay between the properties of \( \top \) and the behavior of \( S_n(X; \top) \) in the CLT.

This inserts in a larger research program to show the relationship between the algebraic properties of a t-norm and the geometry of \( \mathcal{F}_c \) with the corresponding t-norm addition, and its consequences on limit theorems.

We can also distinguish a theoretical and a pragmatic variant of the problem of obtaining a positive CLT for random u.s.c. functions.

Problem 3a. For \( E \) as general as possible, find a family \( \mathcal{F}_c[\top] \subset \mathcal{F}_c \) with the following properties:

(A) The mapping \( \top \mapsto \mathcal{F}_c[\top] \) is not ad hoc, rather it depends on general properties of \( \top \) that provide some insight about Problem 2.

(B) \( \mathcal{F}_c[\min] = \mathcal{F}_c \).

(C) The CLT holds for \( S_n(X; \top) \) if \( X \) is almost surely \( \mathcal{F}_c[\top] \)-valued.

Namely, find sufficient conditions to prove for arbitrary t-normed sums, that the \( \top \)-CLT holds in a ‘nice’ subspace. Since the CLT holds generally for the choice \( \top = \min \), we must demand that in this case every element of \( \mathcal{F}_c \) be considered ‘nice’.

Problem 3b. For \( E = \mathbb{R} \), for families \( G \subset \mathcal{F}_c \) which are commonly used in practice, and at least for relevant examples of t-norms commonly used in practice, show whether the CLT holds or not.

3 Specific results to be presented at the EYSM

Problem 1 will be solved. Concerning Problem 2, we will present some preliminary work that shows that non-classical behaviours are possible. In particular, we will exhibit a random u.s.c. function \( X \) such that:

- For \( a \top b = \min\{a, b\} \), there is central convergence (the CLT holds).
- For \( a \top b = ab \), there is divergence.
- For \( a \top b = \max\{a + b - 1, 0\} \) (Łukasiewicz t-norm), there is non-central convergence (a non-central limit theorem holds).

Concerning Problem 3a, there is promising work ongoing but no definitive result ready to be shown. Hence we concentrate on Problem 3b, which we will solve in the affirmative (in the sense of \( d_p \) convergence) for triangular or trapezoidal functions when \( \top \) is either the product or the Łukasiewicz t-norm. This is only a modest step when compared to the ambitious aim of solving Problem 3a, but from a pragmatical point of view those are by large the most commonly used function shapes and t-norms.
References


