Skeletonization on Projection Pictures

Attila Fazekas

Kossuth Lajos University, Debrecen P.O.Box 12, H-4010, Hungary

Abstract. The template method is one of the most often used group of image transformations in digital image processing. In this paper we try to reveal the possibility of template methods on the perpendicular projections of 2D binary pictures. We examined the theoretical background of the template methods on projection pictures. As a result of this we created an algorithm, which accomplishes this template method. To illustrate the theoretical results, a special template method, the thinning method is implemented.

Keywords. Skeletonization, template method, projection picture

1 Introduction

One of the most often used group of image transformations in digital image processing is the so called template method. This method is commonly used both in preprocessing and for obtaining specific features.

This is especially true in the case of 2D binary pictures. To prove this statement let us refer to the thinning templates, which are used for skeletonization. The literature on this topic is quite rich [6,10]. Besides the thinning on the classical pixel-pixel type picture representations, investigations also include the picture representation based on the quadtree [2], run-length [8], and contour-coding [1,5].

It often happens in medical digital image processing that instead of the digital pictures only their projections are known. These projections are considered as the representations of the original pictures if the unique reconstructibility is assured. The following question arises: if the projections are considered as representations, how can the template methods, especially the thinning method, be applied on these picture representations.

This paper deals with the possibility of template methods on the perpendicular projections of 2D binary pictures. We present the theoretical background of the template method on projection pictures, and as a result of this an algorithm has been created, which provides template methods on projection pictures. In the last chapter we implement a special template method, the thinning method, to illustrate the theoretical results.
This chapter contains some well-known concepts of digital topology which will be used later on.

**Definition 1.** A set \( X \subseteq \mathbb{Z}^2 \) is called a digital set and its elements are called points. A function \( f \) defined on a digital set \( X \) with values \( \{0, 1, \ldots, n - 1\} \) \((n \in \mathbb{N})\) is called an \( n \)-valued digital picture. If \( n = 2 \) then \( f \) is a binary picture.

**Definition 2.** Let \( f \) be a binary picture defined on the digital set \( X \). The set \( F = \{p \mid f(p) = 1, p \in X\} \) is the foreground or object of the binary picture \( f \) and the set \( B = \{p \mid f(p) = 0, p \in X\} \) is the background of \( f \).

**Definition 3.** Let \( X \) be a digital set and \( p(x, y) \in X \) its arbitrary element. The \( N_4(p) \) 4-neighbourhood of the point \( p \) are those points of \( X \) whose \((x', y')\) coordinates satisfy the equality \(|x - x'| + |y - y'| = 1\). The \( N_8(p) \) 8-neighbourhood of the point \( p \) are those points of \( X \) which differ from \( p \) and whose coordinates satisfy the following two inequalities at the same time: \(|x - x'| \leq 1, |y - y'| \leq 1\).

**Definition 4.** Let \( p \) and \( q \) be two points of the digital set \( X \). The \( n \)-path from \( p \) to \( q \) is defined as the sequence of points \( p = p_0, p_1, \ldots, p_k = q \) \((k \in \mathbb{N})\), where the elements of the sequence belong to \( X \) and \( p_i \) is the \( n \)-neighbour of \( p_{i-1} \) \((1 \leq i \leq k)\).

**Definition 5.** A digital set \( X \) is \( n \)-connected if there is an \( n \)-path between any two points \( p \) and \( q \) in \( X \).

**Definition 6.** The "\( n \)-path from \( p \) to \( q \)" relation is an equivalence relation on the digital set \( X \) and the classes induced by this are called the \( n \)-components of \( X \).

**Definition 7.** Let \( p' \) and \( p'' \) be any two points of the \( \mathbb{Z}^2 \), so that \( \Pr_i(p') \leq \Pr_i(p'') \) \((i = 1, 2)\). The \( i \)th coordinate of the point \( p \) is indicated by \( \Pr_i(p) \). The digital set \( W \) is called a window with size \((\Pr_1(p'') - \Pr_1(p') + 1) \times (\Pr_2(p'') - \Pr_2(p') + 1)\) if

\[
W = \{p \mid \Pr_i(p') \leq \Pr_i(p) \leq \Pr_i(p'')\}.
\]
3 Reconstructibility of Binary Matrices

Let \( f \) be a binary picture defined on the window \( X \) with size \( m \times n \). In this case \( f \) can be represented as a binary matrix, element \((i, j)\) of which is equal to 1 if \( f((k + i - 1, l + j - 1)) = 1 \) holds and 0 otherwise, where \( k \) is the horizontal and \( l \) is the vertical coordinate of the upper left corner.

This pixel-pixel type of binary picture representation allows us to characterize those cases when the perpendicular projections of the binary pictures can be considered the representation of the original pictures. In this chapter we are going to investigate it with the help of the results on reconstructibility of binary matrices.

**Definition 8.** Let \( X = (x_{ij}) \) be a binary matrix with size \( m \times n \). On the row projection of the matrix \( X \) we mean the vector \( R(X) = R = (r_1, r_2, \ldots, r_m) \) and on the column projection we mean the vector \( C(X) = C = (c_1, c_2, \ldots, c_n) \), where

\[
    r_i = \sum_{j=1}^{n} x_{ij}, \quad c_j = \sum_{i=1}^{m} x_{ij}.
\]

The class of binary matrices with row and column projections \( R \) and \( C \) is denoted by \( \mathcal{X}(R, C) \).

**Definition 9.** We say that a matrix \( X \) can be uniquely reconstructed from its projections if \( X' \in \mathcal{X}(R, C) \) implies \( X = X' \).

**Theorem 10.** (see in [11]). A binary matrix \( X \in \mathcal{X}(R, C) \) can be uniquely reconstructed if and only if it does not contain switching components \( X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) as a minor.

Using this theorem we can see that the perpendicular projections of binary pictures defined on a window \( X \) can be considered as the reconstruction of the picture \( f \) if and only if the binary matrix can be uniquely reconstructed from its projections with the pixel-pixel picture representation defined by \( f \).

It is possible to generalize the unique reconstruction problem. In this case we can prescribe the elements of the uniquely reconstructible binary matrix.

**Definition 11.** Let \( P = (p_{ij}) \) and \( Q = (q_{ij}) \) be two binary matrices with size \( m \times n \). We say \( Q \succeq P \) if \( q_{ij} \geq p_{ij} \) for all positions \((i, j) \in \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}\). The class of \( \mathcal{X}^Q_P(R, C) \) then can be defined as:

\[
    \mathcal{X}^Q_P(R, C) = \{ X \mid X \in \mathcal{X}(R, C), \ P \leq X \leq Q \}.
\]

It is clear that if \( P = \{0_{ij}\} \) (zero matrix) and \( Q = \{1_{ij}\} \) then we have the class \( \mathcal{X}(R, C) \) again. It is easy to prove [4] that instead of investigating the class
\(X^Q_0(R, C)\), as a consequence of the transformation \(X^Q_{0,P}(R - R(P), C - C(P))\), it is sufficient to investigate \(X^Q_{0,P}(R - R(P), C - C(P))\), which is then shortly called class \(X^{Q - P}(R - R(P), C - C(P))\).

In this class the switching chain plays the role of the switching component [4].

**Definition 12.** We say that the binary matrix \(X \in X^Q(R, C)\) has a switching chain if there is a sequence of different free positions of \(X\),

\[
\{(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \ldots, (i_p, j_p), (i_p, j_1)\},
\]

such that

\[
x_{i_1j_1} = x_{i_2j_2} = \ldots = x_{i_pj_p} = 1 - x_{i_1j_2} = 1 - x_{i_2j_3} = \ldots = 1 - x_{i_1j_1},
\]

\((p \geq 2)\). The position \((i, j)\) is said to be free if the corresponding matrix element is not prescribed by \(Q\), i.e. \(q_{ij} = 1\).

According to this, the unique reconstructibility is characterized as it follows.

**Theorem 13.** (see in [4]) A binary matrix \(X \in X^Q(R, C)\) can be uniquely reconstructed if and only if \(X\) has no switching chain.

#### 4 The Theoretical Background of the Template Methods on Projection Pictures

For template processes on projections of uniquely reconstructable binary matrices, first we have to provide, that the fitting of a template should be determined only by analysing the projections of the template and of the original picture. The following theorem provides the theoretical background.

**Theorem 14.** Let \(X = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix}\), \(P = \begin{pmatrix} P_1 & P_2 & P_3 \\ P_4 & P_5 & P_6 \\ P_7 & P_8 & P_9 \end{pmatrix}\), and \(Q = \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ Q_4 & Q_5 & Q_6 \\ Q_7 & Q_8 & Q_9 \end{pmatrix}\) be binary matrices with size \(m \times n\). Let \(X_i, P_i\), and \(Q_i\) \((i = 1, 2, \ldots, 9)\) be submatrices with the same size in \(X, P,\) and \(Q\), respectively.

Furthermore, let \(\overline{X} = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & \xi_{i} & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix}\), \(X^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_5 & 0 \\ 0 & 0 & 0 \end{pmatrix}\), and \(X^{0'} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & X'_{i} & 0 \\ 0 & 0 & 0 \end{pmatrix}\) be binary matrices with size \(m \times n\). Let \(X_5\) and \(X'_{i}\) be binary matrices with the same size. The projections of binary matrices \(X, X^0, X^{0'}\), and
$X^{0r}$ are denoted by $R(X), R(\overline{X}), R(X^0)$, and $R(X^{0r})$ and $C(X), C(\overline{X}), C(X^0)$, and $C(X^{0r})$. We assume that $X$ is uniquely reconstructible in the class $\mathcal{X}_P^Q$, and $X'_5$ is a non-zero uniquely reconstructible matrix. A matrix $X$ with size $m \times n$ is uniquely reconstructible from projections $R(X) - R(X^{0r})$ and $C(X) - C(X^{0r})$ in the class $\mathcal{X}_P^Q$ if and only if $X'_5 \leq X'_5$, where $\overline{Q} = \begin{pmatrix} Q_1 & Q_2 & Q_3 \\ Q_4 & 0 & Q_6 \\ Q_7 & Q_8 & Q_9 \end{pmatrix}$ and $\overline{P} = \begin{pmatrix} P_1 & P_2 & P_3 \\ P_4 & 0 & P_6 \\ P_7 & P_8 & P_9 \end{pmatrix}$.

Proof of theorem. Suppose that the matrix $X'$ is uniquely reconstructible from projections $R(X') = R(X) - R(X^{0r})$ and $C(X') = C(X) - C(X^{0r})$ in the class $\mathcal{X}_P^Q$. Since the matrix $X, \overline{X}, X^0$ and $X^{0r}$ uniquely reconstructible from their projections, the following equations hold:

$$
R(X') = R(X) - R(X^{0r})
$$
$$
C(X') = C(X) - C(X^{0r})
$$
$$
R(X) = R(\overline{X}) + R(X^0)
$$
$$
C(X) = C(\overline{X}) + C(X^0)
$$

From these equations we have

$$
X = X' + X^{0r}
$$
$$
X = \overline{X} + X^0.
$$

According to the construction of the matrix $X^{0r}$ and $X^{0r}$ if some element of the matrix $\overline{X}$ is equal to 1, then the corresponding element of the matrix $X'$ is also equal to 1. This feature of the matrix $\overline{X}$ and $X'$ implies that the matrix $X' - \overline{X}$ is a binary matrix. This result and the above equations imply that the matrix $X^0 - X^{0r}$ is also a binary matrix. As a consequence, if one of the elements of the matrix $X^{0r}$ is 1, than the corresponding element of the matrix $X^0$ is also 1. Thus $X^{0r} \leq X^0$, so $X'_5 \leq X'_5$.

Conversely. We assume that $X'_5 \leq X'_5$. It is easy to see, that the matrix $X' = X - X^{0r}$ is a binary matrix. We prove the fact that the reconstruction of the matrix $X'$ from projections $R(X) - R(X^{0r})$ and $C(X) - C(X^{0r})$ is unique.

Suppose that the matrix $X'' \neq X'$ can be reconstructed from the projections $R(X) - R(X^{0r})$ and $C(X) - C(X^{0r})$. Then the equations $X' = X - X^{0r}$ and $X'' = X - X^{0r}$ imply the equation $X' + X^{0r} = X'' + X^{0r}$. Since the matrix $X^{0r}$ is uniquely reconstructible from its projections, the equation $X' = X''$ also holds. This is a contradiction. The uniquely reconstruction of the matrix $X'$ in the class $\mathcal{X}_P^Q$ also holds. In order to prove this, it is enough to realize that the matrices $X - X^{0r}$ and $X - X^{0r}$ can only differ in those elements which have not been prescriibed in matrices $\overline{P}$ and $\overline{Q}$.  \[ \square \]
In the following step we are going to determine how the unique reconstructability can be held during template processing. This theorem has been published in paper [3] as a joint work with G.T. Herman and S. Matej.

**Theorem 15.** Let \( X = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & x & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix} \) and \( X' = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & x' & X_6 \\ X_7 & X_8 & X_9 \end{pmatrix} \) are binary matrices, furthermore let \( x \) be the element of matrix \( X \) with coordinates \((i, j)\). Let the equation \( x' = 1 - x \) be true. If the binary matrix \( X \) is uniquely reconstructible in class \( \mathcal{X}^Q_{P} \) then \( X' \) can be uniquely reconstructed in the class \( \mathcal{X}^{Q'}_{P'} \), where \( P' \) and \( Q' \) differ from binary matrices \( P \) and \( Q \) only with the element with coordinates \((i, j)\), which can be defined as \( p_{ij} = q_{ij} = x' \).

**Proof of theorem.** It is easy to verify that in a uniquely reconstructible matrix \( X \), in class \( \mathcal{X}^Q_{P} \), it can happen that by changing one element a switching chain can be constructed, which can damage the uniqueness of the reconstructibility. In order to avoid this the new value of the element changed should be fixed in the matrices \( P \) and \( Q \). This means that the matrix \( X' \) can be uniquely reconstructed in class \( \mathcal{X}^{Q'}_{P'} \).

\( \square \)

5 Projection Template Processing

On the basis of our result we can construct a template method on projection pictures. The idea is the following: First we investigate the fitting of the template in a point of the picture using Theorem 14. Using the notations of this theorem, the fact that there is a matrix, which is uniquely reconstructible from the projections \( R(X) - R(X^{0'}) \) and \( C(X) - C(X^{0'}) \) in the class \( \mathcal{X}^Q_{P} \), means that all the 1's found in the template \( X'_5 \) fit in the 1's in \( X_5 \). Then we check, whether there is such a matrix which can be uniquely reconstructed from the projections \( R(X^c) - R(X^{0'c}) \) and \( C(X^c) - C(X^{0'c}) \) in the class \( \mathcal{X}^Q_{P^c} \). If there is, then it means that the 0's found in the template \( X'_5 \) will fit in the 0's in \( X_5 \).

These two investigations together mean the exact fitting of template \( X'_5 \). If the template fits in the point \( ij \) of the binary matrix, then we can modify it as well as the \( j \)th element of \( R \), and the \( i \)th element of \( C \) on the projection pictures. Then we modify the required elements of sets \( P \) and \( Q \) based on the Theorem 15. This will guarantee the unique reconstructibility of matrix.
Now as an example let us investigate an algorithm which executes the template method in the case of $3 \times 3$ templates. We are going to use this algorithm for our thinning algorithm.

Program Thinning:

```
Var P,Q:Array[1..m,0..n] Of Byte;
PC,QC:Array[1..m,0..n] Of Byte; /* The complement of P and Q */
TP,TQ:Array[1..m,0..n] Of Byte; /* Temporary P and Q */
TPC,TQC:Array[1..m,0..n] Of Byte /* Temporary PC and QC */
R,RC:Array[0..m+1] Of Integer; /* The row projections */
C,CC:Array[0..m+1] Of Integer; /* The column projections */
TR,TRC:Array[0..m+1] Of Integer; /* Temporary R and RC */
TC,TCC:Array[0..m+1] Of Integer; /* Temporary C and CC */
RM,RMC:Array[0..l+1] Of Integer;
CM,CMC:Array[0..m+1] Of Integer;
SR:Array[1..k,1..3] Of Integer; /* Row projections of the templates */
SC:Array[1..k,1..3] Of Integer; /* The same for columns */
SRC:Array[1..k,1..3] Of Integer;
SCC:Array[1..k,1..3] Of Integer;
Fit:Boolean;
Begin
R[0]:=0;R[m+1]:=0;C[0]:=0;C[n+1]:=0; /* Set the border */
RC[0]:=1;RC[m+1]:=1;CC[0]:=1;CC[n+1]:=1; /* Set the border's complement */
Fit:=True;
While Fit Do Begin
    Fit:=False;
    For h:=1 To k Do Begin
        TR:=R;TC:=C;TRC:=TR;TCC:=TC;
        RM:=R;CM:=C;RMC:=RC;CMC:=CC;
        For i:=1 To m Do For j:=1 To l Do Begin
            TR:=R;TC:=C;TRC:=TR;TCC:=TC;
            R[j-1]:=TR[j-1]-SR[h,1];RC[j-1]:=TRC[j-1]-SRC[h,1];
            R[j]:=TR[j]-SR[h,2];RC[j]:=TRC[j]-SRC[h,2];
            R[j+1]:=TR[j+1]-SR[h,3];RC[j+1]:=TRC[j+1]-SRC[h,3];
            C[i-1]:=TC[i-1]-SC[h,1];CC[i-1]:=TCC[i-1]-SCC[h,1];
            C[i]:=TC[i]-SC[h,2];CC[i]:=TCC[i]-SCC[h,2];
            C[i+1]:=TC[i+1]-SC[h,3];CC[i+1]:=TCC[i+1]-SCC[h,3];
            If (R,C) uniquely reconstructible in $\mathcal{A}_{TP}$ Then
                If (RC,CC) uniquely reconstructible in $\mathcal{A}_{TQC}$ Then
                    Begin
                        Fit:=True;
                        P[i,j]:=0;Q[i,j]:=0;PC[i,j]:=1;QC[i,j]:=1;
                        RM[j-1]:=RM[j-1]-1;CM[i-1]:=CM[i-1]-1;
                        RMC[j+1]:=RMC[j+1]+1;CMC[i+1]:=CMC[i+1]+1;
                    End;
            End;
            R:=TR;C:=TC;RC:=TRC;CC:=TCC;
            End;
            R:=RM;C:=CM;RC:=RMC;CC:=CMC;
            End;
            End;
        End;
    End;
End;
```

This algorithm deletes a point of a binary picture if it fits on at least one template. This subcycle should be applied as long as there are any changes on the picture. If we use templates during the thinning procedure, which belong to two or more subcycles then the algorithm should be used with the templates belonging to the adequate subcycle. If we would like to delete every point in a subcycle at the same time on which at least one template fits then we have to
use two picture planes avoiding the case when a previously deleted point would change the neighbours of the examined point.

For thinning we can use the templates in [9]. It is easy to see that these templates fulfill the requirements of being uniquely reconstructible. Fig. 1a shows the original object on which the positions "●" represent the "1" elements, the "○" represent the "0" elements, while the "•" represent those "0" elements which are prescribed to get unique reconstructibility. On Fig. 1b the skeleton is marked with "●", and the positions marked by number i are deleted in the ith subcycle.

![Fig. 1a. The original image.](image1.png)

![Fig. 1b. The skeleton of the original image.](image2.png)

References

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