Geometry of Neighbourhood Sequences

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Abstract

In this paper we generalize some former results of Das and Chatterji [2] about geometric properties of 2-dimensional periodic neighbourhood sequences. We use a more general definition of neighbourhood sequences, which does not require periodicity. As an extension of the former 2D results, we study the geometric behavior of general neighbourhood sequences in arbitrary finite dimension. Since 2D and 3D digital spaces are the most important in digital geometry and in digital image processing, we study these spaces in detail.

Key words:
Digital Geometry, Neighbourhood Sequences
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1 Introduction

The classical digital – cityblock and chessboard – motions were introduced by Rosenfeld and Pfaltz [6]. The cityblock motion allows movements only in horizontal and vertical directions, while the chessboard motion in diagonal directions, as well. Based on these two types of motions, the authors in [6] defined two distances. The $d_4$ or $d_8$ distance of two points is the number of steps required to reach either point from the other, where only cityblock or chessboard motions can be used, respectively. To obtain a better approximation for the Euclidean distance, Rosenfeld and Pfaltz recommended the alternate use of the cityblock and chessboard motions, which defines the distance $d_{oct}$. Geometrically, the corresponding ”disks” are diamonds for the distance $d_4$, squares for $d_8$, and octagons for $d_{oct}$. Hence, $d_{oct}$ provides the best approximation of the Euclidean distance out of these three distances.
By allowing arbitrary mixture of cityblock and chessboard motions, Das et al. introduced the concept of periodic neighbourhood sequences in [1]. In this paper only 2D periodic neighbourhood sequences were investigated. The concept of neighbourhood sequences was extended to arbitrary finite and infinite dimensions in [3]. Moreover, the concept of generalized (not necessarily periodic) neighbourhood sequences was also introduced in [3].

The main advantage of neighbourhood sequences over the classical distances $d_4$, $d_8$ and $d_{a\alpha}$ is that they provide more flexibility in moving on the plane. Making use of this property, Das and Chatterji [2] were able to determine distance functions that provide good approximation of the Euclidean distance. The authors in [2] investigated the quality of the approximation of the Euclidean distance by a periodic neighbourhood sequence. To this purpose they analyzed the geometric properties of the octagons occupied by a neighbourhood sequence during "spreading" on the 2D plane. These geometric investigations were performed only for periodic neighbourhood sequences, and only in $\mathbb{Z}^2$. Using generalized neighbourhood sequences (which are not required to be periodic) Euclidean distance can be approximated more precisely. For this reason, we extend the geometric description of periodic sequences to generalized neighbourhood sequences, and embed the former results into this more general structure. Moreover, we perform our analysis in $\mathbb{Z}^n$ instead of $\mathbb{Z}^2$. Since 2D and 3D digital spaces are the most important in digital geometry and in digital image processing, we study these spaces in detail.

The structure of this paper is as follows. In the second section we give our notation, and provide some properties of the concepts introduced. In the third section we extend the former 2D results of Das and Chatterji [2] to the $n$D digital space $\mathbb{Z}^n$. Moreover, we use generalized neighbourhood sequences instead of periodic ones in our analysis. In the fourth section we give a detailed and illustrated description of the geometric behavior of neighbourhood sequences in $\mathbb{Z}^2$ and $\mathbb{Z}^3$. In the last section we summarize our results.

2 Notation and basic concepts

In this section we recall some definitions and notation from [1] and [3] concerning neighbourhood sequences. In what follows, $n$ denotes a positive integer.

Let $p$ be a point in $\mathbb{Z}^n$. The $i$-th coordinate of $p$ is indicated by $Pr_i(p)$ ($1 \leq i \leq n$). Let $M$ be an integer with $0 \leq M \leq n$. The points $p, q \in \mathbb{Z}^n$ are called $M$-neighbours, if the following two conditions hold:

- $|Pr_i(p) - Pr_i(q)| \leq 1$ \hspace{1em} ($1 \leq i \leq n$),
\[ \sum_{i=1}^{n} |\Pr_i(p) - \Pr_i(q)| \leq M. \]

The sequence \( A = (a(i))_{i=1}^{\infty} \), where \( a(i) \in \{1, \ldots, n\} \) for all \( i \in \mathbb{N} \), is called an \( n \)-dimensional (shortly \( n \text{D} \)) neighbourhood sequence. \( A \) is periodic, if for some \( l \in \mathbb{N} \), \( a(i + l) = a(i) \ (i \in \mathbb{N}) \). For a periodic neighbourhood sequence \( A \) with period \( l \) we briefly write \( A = (a(1), a(2), \ldots, a(l)) \). The set of the \( n \text{D} \)-neighbourhood sequences will be denoted by \( S_n \).

Let \( p, q \in \mathbb{Z}^n \) and \( A \in S_n \). The point sequence \( p = p_0, p_1, \ldots, p_m = q \), where \( p_{i-1} \) and \( p_i \) are \( a(i) \)-neighbours in \( \mathbb{Z}^n \ (1 \leq i \leq m) \), is called an \( A \)-path from \( p \) to \( q \) of length \( m \). The \( A \)-distance \( d(p, q; A) \) of \( p \) and \( q \) is defined as the length of the shortest \( A \)-path(s) between them.

In this paper we investigate the way a neighbourhood sequence spreads in the digital space starting from a point of \( \mathbb{Z}^n \). This spreading is translation-invariant, so for simplicity we may choose the origin \( 0 \) of \( \mathbb{Z}^n \) as the starting point.

Let \( A \) be an \( n \text{D} \)-neighbourhood sequence. For \( k \in \mathbb{N} \), let

\[ A_k = \{ p \in \mathbb{Z}^n : d(0, p; A) \leq k \}. \]

So \( A_k \) is the region occupied by \( A \) after \( k \) steps. Let \( H(A_k) \) be the convex hull of \( A_k \) in \( \mathbb{R}^n \).

**Remark 1** We note that \( A_k \) is digitally convex in the usual sense (see, e.g., p. 171, Definition 4.3.4. in [7]).

The following remarks summarize some simple observations about the geometric behavior of neighbourhood sequences.

**Remark 2** For any \( k \in \mathbb{N} \), the region \( H(A_k) \) does not depend on the order of the first \( k \) elements of \( A \).

**Remark 3** For any \( A \in S_n \), the sequence of regions \( (H(A_k))_{k=1}^{\infty} \) is a strictly monotone increasing sequence. That is, \( l > k \) implies \( H(A_l) \supseteq H(A_k) \).

In our investigations we need an explicit formula to calculate the \( A \)-distance of two points in \( \mathbb{Z}^n \), where \( A \in S_n \). The following lemma provides such a tool. Before formulating the lemma, we need to introduce some further notation.

Let \( p \) and \( q \) be two points in \( \mathbb{Z}^n \), and \( A = (a(i))_{i=1}^{\infty} \in S_n \). Let

\[ \|p - q\| = \sum_{k=1}^{n} |\Pr_k(p) - \Pr_k(q)|, \]
and for every $i \in \mathbb{N}$ and $j = 1, \ldots, n$ put

$$a_j(i) = \min(a(i), j), \quad \text{and} \quad f_j^A(i) = \sum_{k=1}^{i} a_j(k).$$

Furthermore, let

$$x = (x(1), x(2), \ldots, x(n))$$

be the nonincreasing ordering of $|\Pr_i(p) - \Pr_i(q)|$ $(1 \leq i \leq n)$, that is, $x(i) \geq x(j)$ if $i < j$. For $k = 1, \ldots, n$ put

$$b_k = \sum_{j=1}^{n-k+1} x(j).$$

Das et al. in [1] provided an algorithm to calculate $d(p, q; A)$, where $A$ is a periodic $n$D-neighbourhood sequence. By the following lemma we extend this result to any neighbourhood sequence belonging to $S_n$.

**Lemma 4** Let $p$ and $q$ be two points in $\mathbb{Z}^n$, and $A \in S_n$. Write $c = ||p - q||$, and let

$$g_k(i) = f_k(c) - f_k(i - 1) - 1, \quad 1 \leq i \leq c.$$

Then the $A$-distance of $p$ and $q$ can be calculated as

$$d(p, q; A) = \max_{1 \leq k \leq n} d_k(p, q),$$

where

$$d_k(p, q) = \sum_{i=1}^{c} \left\lfloor \frac{b_k + g_k(i)}{f_k(c)} \right\rfloor.$$

**Proof.** As we mentioned, a similar statement has been formulated by Das et al. in [1]. They showed that in case of a periodic neighbourhood sequence with period $l$ the statement of the lemma holds with

$$g_k(i) = f_k(l) - f_k(i - 1) - 1 \quad (1 \leq i \leq l),$$

and

$$d_k(p, q) = \sum_{i=1}^{l} \left\lfloor \frac{b_k + g_k(i)}{f_k(l)} \right\rfloor.$$

Suppose that $A$ is any neighbourhood sequence. It is obvious that to calculate the $A$-distance of the points $p, q \in \mathbb{Z}^n$, it is enough to deal with the first $c$ elements of $A$, since $d(p, q; A) \leq c$. In other words, the elements of $A$ after the $c$-th element are not involved into the calculation of $d(p, q; A)$. So if we consider the neighbourhood sequence $A$ as if it was a neighbourhood sequence
with period \( c \) (changing its elements \( a(i) \) with \( i > c \) appropriately), then by applying the result in [1], the proof is complete. \( \Box \)

3 Geometric properties of \( n \)D-neighbourhood sequences

We start our geometric investigations in the general digital space \( \mathbb{Z}^n \). Since neighbourhood sequences spread in an "isotropic" way, the occupied regions are symmetric objects. More precisely, we have the following theorem.

**Theorem 5** Let \( A \in S_n \) and \( k \in \mathbb{N} \). If a point \( p \in \mathbb{Z}^n \) with coordinates \((p_1, p_2, \ldots, p_n)\) belongs to \( A_k \), then the points with coordinates

\[
(\delta_1 p_1, \delta_2 p_2, \ldots, \delta_n p_n)
\]

also belong to \( A_k \). Here \( \delta_j = \pm 1 \) (\( j = 1, \ldots, n \)), and \((i_1, i_2, \ldots, i_n)\) is an arbitrary permutation of \((1, 2, \ldots, n)\).

**Proof.** Suppose that \( p \in A_k \) with coordinates \((p_1, p_2, \ldots, p_n)\). Let \( p' \in \mathbb{Z}^n \) with coordinates \((\delta_1 p_1, \delta_2 p_2, \ldots, \delta_n p_n)\), where \( \delta_i = \pm 1 \) (\( i = 1, \ldots, n \)), and \((i_1, i_2, \ldots, i_n)\) is an arbitrary permutation of \((1, 2, \ldots, n)\). Observe that for every \( k \) with \( 1 \leq k \leq n \), the same \( b_k \) are obtained in Lemma 4 for both \((0, p)\) and \((0, p')\), and \( |0, p| = |0, p'| \). Using this lemma we get \( d(0, p; A) = d(0, p'; A) \), which completes the proof. \( \Box \)

**Remark 6** It is easy to verify that the above theorem also holds for the points of \( H(A_k) \).

Using the above results we can find hyperplanes, for which the regions occupied by neighbourhood sequences are symmetric.

**Remark 7** Let \( A \in S_n \) and \( k \in \mathbb{N} \). Then \( H(A_k) \) is symmetric to every \((n - 1)D\) hyperplanes that contain \((n - 1)\) coordinate axes (this implies that the coordinate values can change sign), and to their rotations by \(45^\circ\) around every axis they contain (this implies that coordinates can be permuted).

Let \( k \in \mathbb{N} \) and let \( k(i) \) denote the number of \( i \) values \((1 \leq i \leq n)\) among the first \( k \) elements of \( A \). First we calculate the coordinates of vertices of polyhedra occupied by neighbourhood sequences.

**Theorem 8** Let \( A \in S_n \) and \( k \in \mathbb{N} \). The vertices of \( H(A_k) \) are exactly those points, whose coordinates are the permutation of the values

\[
\left( \delta_1 \sum_{i=1}^{n} k(i), \delta_2 \sum_{i=2}^{n} k(i), \ldots, \delta_n k(n) \right),
\]

5
where \( \delta_j = \pm 1 \) for every \( j = 1, \ldots, n \), and \( \delta_j \) can be different in different permutations.

**Proof.** We prove the theorem by induction. In 2D Das and Chatterji in [2] showed that the coordinates of the vertices of the occupied regions are the permutation of the following one

\[
(\delta_1(k(1) + k(2)), \delta_2 k(2), \ldots)
\]

where \( \delta_j = \pm 1 \) \( (j = 1, \ldots, n) \).

Now suppose that for any \( H(A_k) \) in \( \mathbb{Z}^n \) the coordinates of the vertices are the permutations of

\[
\left( \delta_1 \sum_{i=1}^{n} k(i), \delta_2 \sum_{i=2}^{n} k(i), \ldots, \delta_n k(n) \right),
\]

with \( \delta_j = \pm 1 \) \( (j = 1, \ldots, n) \). Let \( B \in S_{n+1} \) and \( k \in \mathbb{N} \). The projection of \( H(B_k) \subseteq \mathbb{Z}^{n+1} \) into \( \mathbb{Z}^n \) can be obtained by changing the \((n+1)\)-th coordinate to 0. Moreover, since in \( \mathbb{Z}^n \) there are only \( n \) coordinates, in determining the projection the values \((n+1)\) in \( B \) act as the values \( n \). Thus, by the induction hypothesis the coordinates of the vertices of the projection of \( H(B_k) \) are

\[
\left( \delta_1 \sum_{i=1}^{n+1} k(i), \delta_2 \sum_{i=2}^{n+1} k(i), \ldots, \delta_n \sum_{i=n}^{n+1} k(i), 0 \right),
\]

where the first \( n \) coordinates can be permuted arbitrarily with \( \delta_j = \pm 1 \) \( (j = 1, \ldots, n) \).

Now to get the vertices of \( H(B_k) \) in \( \mathbb{Z}^{n+1} \) we should choose the ”maximal” value for the \((n+1)\)-th coordinate in \((*)\). It means that the absolute value of the \((n+1)\)-th coordinate must be \( k(n+1) \). The symmetric behavior of \( H(B_k) \) implies that the coordinates of the vertices of \( H(B_k) \) are the permutations of the coordinates

\[
\left( \delta_1 \sum_{i=1}^{n+1} k(i), \delta_2 \sum_{i=2}^{n+1} k(i), \ldots, \delta_{n+1} k(n + 1) \right),
\]

with \( \delta_j = \pm 1 \) \( (j = 1, \ldots, n + 1) \), and the proof is complete. \( \square \)

**Remark 9** Using the above theorem we obtain that the maximal number of vertices of \( H(A_k) \) is \( 2^n \cdot n! \). \( H(A_k) \) can be degenerate if some of the elements \((1, \ldots, n)\) do not occur in \( A \), causing the decrease of the number of the vertices.

We investigate the existence of some convergence limits related to the sequence of regions \( (H(A_k))_{k=1}^\infty \). In the general \( n \)-dimensional case only the asymptotic
behaviour of the diameter of $H(A_k)$ is considered, where we use the usual definition of diameter, that is $diam(H(A_k)) = \sup_{p,q \in H(A_k)} e(p,q)$, where $e$ denote the Euclidean metric. We use the densities of the $(1, \ldots, n)$ elements in $A$ to calculate the corresponding limit.

**Definition 10** Let $A \in S_n$. The density of the value $i$ ($i = 1, \ldots, n$) in $A$ is

$$D(i) = \lim_{k \to \infty} \frac{k(i)}{k},$$

if this limit exists.

**Remark 11** If $D(i)$ exists for every $i = 1, \ldots, n$, then $\sum_{i=1}^{n} D(i) = 1$.

**Remark 12** If $A \in S_n$ is a periodic neighbourhood sequence, then $D(i)$ exists for every $i = 1, \ldots, n$, and it is a rational number. More precisely, $D(i)$ is the ratio of the number of $i$ values inside a period.

**Theorem 13** Let $A \in S_n$, and suppose that $D(i)$ exists for every $i = 1, \ldots, n$. Then

$$\lim_{k \to \infty} \frac{diam(H(A_k))}{k} = 2 \left( \sum_{i=1}^{n} iD(i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} iD(i)D(j) \right)^{\frac{1}{2}}.$$ 

**PROOF.**

By Remark 7 and Theorem 8, $diam(H(A_k))$ is the Euclidean distance of the points

$$\left( \sum_{i=1}^{n} k(i), \sum_{i=2}^{n} k(i), \ldots, k(n) \right), \text{ and } \left( -\sum_{i=1}^{n} k(i), -\sum_{i=2}^{n} k(i), \ldots, -k(n) \right).$$

We obtain

$$\lim_{k \to \infty} \frac{diam(H(A_k))}{k} = \lim_{k \to \infty} \frac{2 \left( \sum_{i=1}^{n} ik(i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} ik(i)k(j) \right)^{\frac{1}{2}}}{k} = 2 \left( \sum_{i=1}^{n} iD(i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} iD(i)D(j) \right)^{\frac{1}{2}} . \quad \Box$$
4 Geometric properties of 2D- and 3D-neighbourhood sequences

In this section we perform a detailed investigation of the geometry of neighbourhood sequences in $\mathbb{Z}^2$ and $\mathbb{Z}^3$. We note that our analysis could be done in higher dimensions and also in other kind of digital spaces (e.g., that are based on hexagonal or triangular neighbouring relations).

Similarly to the $n$D case, we use the densities of the elements of the neighbourhood sequences to analyze the asymptotic behaviour of the occupied regions. Since these density values do not always exist, we consider the lower and upper densities, as well.

**Definition 14** Let $A \in S_n$. For $i = 1, \ldots, n$ let

$$s(i) = \liminf_{k \to \infty} \frac{k(i)}{k} \quad \text{and} \quad S(i) = \limsup_{k \to \infty} \frac{k(i)}{k}.$$

**Remark 15** If $s(i) = S(i)$, then $D(i)$ exists, and $D(i) = s(i) = S(i) \ (1 \leq i \leq n)$.

4.1 The 2D case

In [2] Das and Chatterji showed that for every 2D periodic neighbourhood sequence $A$, $H(A_k)$ is always an octagon. They calculated the coordinates of the vertices of the octagon, and also the length of its sides based on the ratio of the 1 and 2 values in a period of $A$. Using generalized neighbourhood sequences we obtain similar results, but in this case we have to involve the densities of the 1 and 2 values in the neighbourhood sequence.

The well-known cityblock and chessboard motions can be represented by the neighborhood sequences $(1, 1, 1, 1, \ldots)$ and $(2, 2, 2, 2, \ldots)$, respectively. In these cases we also obtain ”octagons” after every step, though they become degenerate as it can be seen in Figure 1. In this figure we show a ”typical” octagon occupied by the neighbourhood sequence $(1, 2, 1, 2, \ldots)$, as well.

**Remark 16** If we start any 2D-neighbourhood sequence $A$ from the origin, then for every $k \in \mathbb{N}$ the octagon $H(A_k)$ is symmetric (by Remark 7) to both coordinate axes, and to their $45^\circ$ rotations.

In what follows, we calculate some geometric parameters of these octagons, namely the length of their sides, their perimeters and areas. We give these data in terms of $k(1)$ and $k(2)$.

**Definition 17** Let $A$ be a 2D-neighbourhood sequence. Let $x(k)$ be the length
Fig. 1. Octagons occupied by the 2D-neighbourhood sequences (a) \((1,1,1,1,\ldots)\), (b) \((1,2,1,2,\ldots)\), (c) \((2,2,2,\ldots)\).

of the horizontal and \(y(k)\) be the length of the inclined sides of the octagon \(H(A_k)\), illustrated in Figure 1. Moreover, let \(P_{2D}(k)\) be the perimeter and \(V_{2D}(k)\) be the area of this octagon.

**Remark 18** Because of symmetry, the lengths of the horizontal and vertical sides of these octagons are equal, and the same holds for the inclined sides.

**Proposition 19** Using the above notation, the following relations hold:

- \(x(k) = 2k(2)\),
- \(y(k) = \sqrt{2} k(1)\),
- \(P_{2D}(k) = 4 \left( \sqrt{2} k(1) + 2k(2) \right)\),
- \(V_{2D}(k) = 2 \left( k(1)^2 + 4k(1)k(2) + 2k(2)^2 \right)\).

**PROOF.** By Theorem 8 we know the coordinates of the vertices of \(H(A_k)\). So the statement follows by a simple calculation. \(\square\)

Our purpose is to describe the asymptotic behaviour of the sequences \(x(k)/k\), \(y(k)/k\), \(P_{2D}(k)/k\), and \(V_{2D}(k)/k^2\). We will do this using the lower and upper densities of the 1 and 2 values in the neighbourhood sequence.

**Theorem 20** Let \(A \in S_2\). Then we have

- \(\liminf_{k \to \infty} \frac{x(k)}{k} = 2(1 - S(1)), \quad \limsup_{k \to \infty} \frac{x(k)}{k} = 2(1 - s(1))\);

- \(\liminf_{k \to \infty} \frac{y(k)}{k} = \sqrt{2} s(1), \quad \limsup_{k \to \infty} \frac{y(k)}{k} = \sqrt{2} S(1)\);

- \(\liminf_{k \to \infty} \frac{P_{2D}(k)}{k} = 8 + (4\sqrt{2} - 8)S(1)\),

\(9\)
\[
\limsup_{k \to \infty} \frac{P_{2D}(k)}{k} = 8 + (4\sqrt{2} - 8)s(1);
\]

\[
\liminf_{k \to \infty} \frac{V_{2D}(k)}{k^2} = 2(2 - S(1)^2),
\]

\[
\limsup_{k \to \infty} \frac{V_{2D}(k)}{k^2} = 2(2 - s(1)^2).
\]

**Proof.** The statements are immediate consequences of Proposition 19 by the identity \(k(1) + k(2) = k\) and the well-known properties of \(\liminf\) and \(\limsup\). \(\square\)

We note that as \(s(1) + S(2) = s(2) + S(1) = 1\), the above \(\liminf\) and \(\limsup\) values can be expressed using \(s(2)\) and \(S(2)\). If the densities \(D(1)\) and \(D(2)\) exist, we can obtain the above limits by their help.

**Corollary 21** Let \(A \in S_2\). If \(D(1)\) exists, then we have

\[
\lim_{k \to \infty} \frac{x(k)}{k} = 2(1 - D(1)),
\]

\[
\lim_{k \to \infty} \frac{y(k)}{k} = \sqrt{2}D(1),
\]

\[
\lim_{k \to \infty} \frac{P_{2D}(k)}{k} = 8 + (4\sqrt{2} - 8)D(1),
\]

\[
\lim_{k \to \infty} \frac{V_{2D}(k)}{k^2} = 2(2 - D(1)^2).
\]

**Remark 22** We can formulate the opposite statement as well. Namely, for any \(A \in S_2\) the density \(D(1)\) exists, if and only if any of the convergence limits formulated in Corollary 21 exist.

### 4.2 The 3D case

We continue our analysis by studying the geometry of neighbourhood sequences in \(Z^3\).

As a consequence of Theorem 8 we obtain that if \(A \in S_3\) and \(k \in \mathbb{N}\), then the region \(H(A_k)\) is a polyhedron with at most \(2^3 \cdot 3! = 48\) vertices, 72 edges and 26 faces. The vertices can be obtained by permuting the coordinates \((\delta_1(k(1) + k(2) + k(3)), \delta_2(k(2) + k(3)), \delta_3(k(3)))\), where \(\delta_i = \pm 1\) (\(i = 1, 2, 3\)).

Moreover, the polyhedron \(H(A_k)\) is symmetric to the planes \((x, y), (y, z)\) and \((x, z)\) and to their 45° rotations around each of the coordinate axes contained.
The symmetry of the polyhedron to these planes can be observed in Figures 2, 3 and 4.

We give some examples to illustrate how 3D-neighbourhood sequences spread in \( \mathbb{Z}^3 \). Beside the general case, we show degenerate polyhedra that belong to special types of neighbourhood sequences. In Figure 2 below, we can see polyhedra occupied by constant neighbourhood sequences.

![Fig. 2. Polyhedra occupied by constant 3D-neighbourhood sequences (a) \((1,1,1, \ldots)\), (b) \((2,2,2, \ldots)\), (c) \((3,3,3, \ldots)\).](image1)

In our next example (see Figure 3) we illustrate the occupied polyhedra of neighbourhood sequences containing only two types of elements.

![Fig. 3. Polyhedra occupied by special 3D-neighbourhood sequences (a) \((1,2,1,2, \ldots)\), (b) \((1,3,1,3, \ldots)\), (c) \((2,3,2,3, \ldots)\).](image2)

Finally, in Figure 4 we show the occupied polyhedron of the 3D-neighbourhood sequence \((1,2,3,1,2,3, \ldots)\) after \(k\) steps. We also indicate the edges, which can be of different lengths in the general case.

Similarly to the 2D case, we calculate some geometric parameters of these polyhedra, namely the length of their sides, their surfaces and volumes. Again, we investigate the existence of some convergence limits related to the sequence of polyhedra \((H(A_k))_{k=1}^{\infty}\). As in 2D, we express these values by the densities of the elements in the neighbourhood sequence \(A\).

The side lengths, surfaces and volumes of these polyhedra can be given in terms of \(k(1)\), \(k(2)\) and \(k(3)\).
Fig. 4. Polyhedron occupied the 3D-neighbourhood sequence \((1, 2, 3, 1, 2, 3, \ldots)\).

**Definition 23** Let \(A \in S_3\). By symmetry, the sides of \(H(A_k)\) are of (at most) three different lengths. They are indicated by \(x(k)\), \(y(k)\) and \(z(k)\), as it is shown in Figure 4. Moreover, let \(P_{3D}(k)\) be the surface and \(V_{3D}(k)\) be the volume of this polyhedron.

**Proposition 24** Using the above notation, the following relations hold:

- \(x(k) = 2k(3)\);
- \(y(k) = \sqrt{2}k(2)\);
- \(z(k) = \sqrt{2}k(1)\);
- \(P_{3D}(k) = 6P_{\text{oct}} + 12P_{\text{rec}} + 8P_{\text{hex}}\), where
  \[P_{\text{oct}} = 2(k(2)^2 + 4k(2)k(3) + 2k(3)^2),\]
  \[P_{\text{rec}} = 2\sqrt{2}k(1)k(3),\]
  \[P_{\text{hex}} = \frac{\sqrt{3}}{2}(k(1)^2 + 4k(1)k(2) + k(2)^2);\]
- \(V_{3D}(k) = 6\frac{P_{\text{oct}}H_{\text{oct}}}{3} + 12\frac{P_{\text{rec}}H_{\text{rec}}}{3} + 8\frac{P_{\text{hex}}H_{\text{hex}}}{3}\), where
  \(H_{\text{oct}} = k(1) + k(2) + k(3),\)
  \(H_{\text{rec}} = \frac{\sqrt{2}}{2}(k(1) + 2k(2) + 2k(3)),\)
  \(H_{\text{hex}} = \frac{\sqrt{3}}{3}(k(1) + 2k(2) + 3k(3)).\)

**Proof.** Similarly to Proposition 19 in the 2D case, the length of the sides can be determined by calculating the Euclidean distance of certain vertices,
whose coordinates can be obtained by Theorem 8. The surface consists of the area of 6 octagons, 12 rectangles and 8 hexagons. The volume can be derived by summarizing the volume of certain pyramids, with octagonal, rectangular and hexagonal bases, respectively. □

Now we study the convergence limits of the sequences \(x(k)/k, y(k)/k, z(k)/k, P_{3D}(k)/k^2\), and \(V_{3D}(k)/k^3\). As in the 2D case, we use the lower and upper density values of the elements in the 3D-neighbourhood sequence to estimate these limits.

**Theorem 25** Let \(A \in S_3\). Then we have

- \(\liminf_{k \to \infty} \frac{x(k)}{k} = 2s(3), \ \limsup_{k \to \infty} \frac{x(k)}{k} = 2S(3)\);

- \(\liminf_{k \to \infty} \frac{y(k)}{k} = \sqrt{2}s(2), \ \limsup_{k \to \infty} \frac{y(k)}{k} = \sqrt{2}S(2)\);

- \(\liminf_{k \to \infty} \frac{z(k)}{k} = \sqrt{2}s(1), \ \limsup_{k \to \infty} \frac{z(k)}{k} = \sqrt{2}S(1)\);

- \(\liminf_{k \to \infty} \frac{P_{3D}(k)}{k^2} \geq 12(s(2)^2 + 4s(2)s(3) + 2s(3)^2) + 24\sqrt{2}s(1)s(3) + 4\sqrt{3}(s(1)^2 + 4s(1)s(2) + s(2)^2)\);

- \(\limsup_{k \to \infty} \frac{P_{3D}(k)}{k^2} \leq 12(S(2)^2 + 4S(2)S(3) + 2S(3)^2) + 24\sqrt{2}S(1)S(3) + 4\sqrt{3}(S(1)^2 + 4S(1)S(2) + S(2)^2)\);

- \(\liminf_{k \to \infty} \frac{V_{3D}(k)}{k^3} \geq 4(s(2)^2 + 4s(2)s(3) + 2s(3)^2) + 8s(1)s(3)(1 + s(2) + s(3)) + \frac{4}{3}(s(1)^2 + 4s(1)s(2) + s(2)^2)(1 + s(2) + 2s(3))\);

- \(\limsup_{k \to \infty} \frac{V_{3D}(k)}{k^3} \leq 4(S(2)^2 + 4S(2)S(3) + 2S(3)^2) + 8S(1)S(3)(1 + S(2) + S(3)) + \frac{4}{3}(S(1)^2 + 4S(1)S(2) + S(2)^2)(1 + S(2) + 2S(3))\).

**Proof.** The statements are immediate consequences of Proposition 24 and the well-known properties of \(\liminf\) and \(\limsup\). □

Contrary to 2D (Theorem 20), in some cases we can have only inequalities for \(\liminf\) and \(\limsup\) in 3D and also in higher dimensions. If the densities of the elements of a neighbourhood sequence exist we get equalities.
Corollary 26 Let $A \in S_3$. If $D(1)$, $D(2)$ and $D(3)$ exist, then we have

- $\lim_{k \to \infty} \frac{x(k)}{k} = 2D(3)$;
- $\lim_{k \to \infty} \frac{y(k)}{k} = \sqrt{2}D(2)$;
- $\lim_{k \to \infty} \frac{z(k)}{k} = \sqrt{2}D(1)$;
- $\lim_{k \to \infty} \frac{P_{\rho(k)}}{k^2} = 12(D(2)^2 + 4D(2)D(3) + 2D(3)^2) + 24\sqrt{2}D(1)D(3) + 4\sqrt{3}(D(1)^2 + 4D(1)D(2) + D(2)^2);$
- $\lim_{k \to \infty} \frac{V_{\alpha(k)}}{k^3} = 4(D(2)^2 + 4D(2)D(3) + 2D(3)^2) + 8D(1)D(3)(1 + D(2) + D(3)) + \frac{4}{3}(D(1)^2 + 4D(1)D(2) + D(2)^2)(1 + D(2) + 2D(3)).$

Remark 27 Similarly to the 2D case, an opposite statement also can be formulated. Namely, for any 3D-neighbourhood sequence the densities $D(1)$, $D(2)$ and $D(3)$ exist if and only if any of the convergence limits concerning any two sides, the surface or the volume in Corollary 26 exist.

5 Conclusion

In this paper we extend and generalize some former geometric results about 2D-neighbourhood sequences. We give the coordinates of vertices of polyhedra occupied by $n$-D-neighbourhood sequences, and perform the symmetry and convexity analysis of these bodies. Since the 2D and 3D digital grids play a very important role in digital image processing, we investigate the digital spaces $Z^2$ and $Z^3$ in detail. In these calculations we use the densities of the elements of neighbourhood sequences. The geometric analysis performed in the paper can be used in approximating Euclidean objects by digital ones. We note that the theory of neighbourhood sequences should not be restricted to $Z^n$, other kind of grids also can be used (see, e.g., [4,5]).

References


