Parameterized Norm Form Equations with Arithmetic progressions

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Abstract

Let $\alpha$ be a zero of the Thomas polynomial $X^3 - (a - 1)X^2 - (a + 2)X - 1$. We find all algebraic numbers $\mu = x_0 + x_1\alpha + x_2\alpha^2 \in \mathbb{Z}[\alpha]$, such that $x_0, x_1, x_2 \in \mathbb{Z}$ forms an arithmetic progression and the norm of $\mu$ is less than $|2a + 1|$. In order to find all progressions we reduce our problem to solve a family of Thue equations and solve this family completely.

Key words: Arithmetic progressions, norm form equations, Thue equations

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1 Introduction

Buchmann and Pethő [5] observed that following algebraic integer

\[ 10 + 9\alpha + 8\alpha^2 + 7\alpha^3 + 6\alpha^4 + 5\alpha^5 + 4\alpha^6, \]

with \( \alpha^7 = 3 \) is a unit. Since the coefficients form an arithmetic progressions they have found a solution to the Diophantine equation

\[ N_{K/Q}(x_0 + \alpha x_1 + \cdots + x_6\alpha^6) = \pm 1, \] (1)

such that \((x_0, \ldots, x_6) \in \mathbb{Z}^7\) is an arithmetic progression.

A full norm form equation is defined by

\[ N_{K/Q}(x_0 + \alpha x_1 + \cdots + x_{n-1}\alpha^{n-1}) = m, \] (2)

where \( \alpha \) is an algebraic integer of degree \( n \), \( K = \mathbb{Q}(\alpha) \), \( m \in \mathbb{Z} \) and \((x_0, x_1, \ldots, x_{n-1}) \in \mathbb{Z}^n \). It is well known that (2) admits infinitely many solutions for infinitely many \( m \) [14]. This is already true for \( m = 1 \). On the other hand Bérczes and Pethő [3] proved that (2) has only finitely many solutions that form an arithmetic progression provided \( \beta := n\alpha^n - \frac{a}{\alpha - 1} \) is an algebraic number of degree at least 3. Moreover they showed that the solution found by Buchmann and Pethő is the only solution to (1).

Bérczes and Pethő also considered arithmetic progressions arising from the norm form equation (2), where \( \alpha \) is a root of \( X^n - a \), with \( n \geq 3 \) and \( 2 \leq a \leq 100 \) (see [2]).

Let \( f_a \in \mathbb{Z}[X], a \in \mathbb{Z} \) be the family of simplest cubic polynomials

\[ f_a := X^3 - (a - 1)X^2 - (a + 2)X - 1. \]

Let \( \alpha = \alpha_a \) be a root of \( f_a \) and put \( K = \mathbb{Q}(\alpha) \). It follows from a result of Lemmermeyer and Pethő [9] that the equation

\[ \left| N_{K/Q}(x_0 + x_1\alpha + x_2\alpha^2) \right| = |m| \] (3)

with \( |m| \leq |2a + 1|, m \in \mathbb{Z} \) has infinitely many solutions \((x_0, x_1, x_2) \in \mathbb{Z}^3 \) if and only if \( m \) is a cube of an integer or \( m = \pm(2a + 1) \). By the above mentioned result of Bérczes and Pethő [3] equation (3) has for every \( a \in \mathbb{Z} \) and \( |m| \leq |2a + 1|, m \in \mathbb{Z} \) only finitely many solutions \((x_0, x_1, x_2) \in \mathbb{Z}^3 \), which form an arithmetic progression.

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The aim of this paper is to describe completely those solutions, which form an arithmetic progression. A solution \((x_0, x_1, x_2) \in \mathbb{Z}^3\) of (3) is called primitive, if \(\gcd(x_0, x_1, x_2) = 1\). With this convention we prove the following theorem:

**Theorem 1** Let \(\alpha\) be a root of the polynomial \(f_a\), with \(a \in \mathbb{Z}\). Then the only solutions to the norm form inequality

\[
\left| N_{K/Q}(x_0 + x_1\alpha + x_2\alpha^2) \right| \leq |2a + 1|
\]  

(4)

such that \(x_0 < x_1 < x_2\) is an arithmetic progression and \((x_1, x_2, x_3)\) is primitive are either \((x_1, x_2, x_3) = (-2, -1, 0), (-1, 0, 1)\) and \((0, 1, 2)\), or they are sporadic solutions that are listed in table 1.

Table 1

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In table 1 we only list solutions, where the parameter is non-negative. Furthermore \(m\) denotes the value of the norm, i.e. \(N_{K/Q}(x_0 + x_1\alpha + x_2\alpha^2) = m\). Lemma 1 will show that it suffices to study the norm inequality (4) only for \(a \geq 0 \in \mathbb{Z}\). Moreover, Lemma 1 gives a correspondence between solutions for \(a\) and \(-a - 1\).

To prove the main Theorem 1 we transform (4) to a parametrized family of Thue inequalities (5). From here on we follow essentially the line of [12]. Although there are a lot of parametrized families of Thue equations and inequalities, which were solved completely, our example (5) admits additional difficulty, because the coefficient of both unknowns depend on \(\alpha\). Therefore we need more precise information on the arithmetic of \(\mathbb{Z}[\alpha]\), especially we need
a basis of its unit group. Fortunately this is known by the result of Thomas [17].

The plan of the proof is as follows. First (Section 2) we show how our problem is connected with a family of Thue inequalities. In order to solve this family we have to do a lot of symbolic computations and we therefore need good approximations to the roots of the relevant polynomial (7) (see Section 3).

The proof of the main Theorem 1 is split into four steps. The first step is to find an upper bound \( a_0 \) for the parameter \( a \) such that there are no further solutions if \( a \geq a_0 \). This bound is found by an application of a variant of Baker’s method combined with technical computations (see Sections 4 and 5). In particular we use linear forms in two logarithms and apply a powerful theorem due to Laurent, Mignotte and Nesterenko [8].

The bound which is found in the previous step is too big to solve all remaining Thue inequalities. We have to consider essentially two different cases (occurring from the linear forms of logarithms used in Section 5). The first case is treated in Section 6 by a method due to Mignotte [11]. For an application of this method we have to reconsider the linear forms treated in Section 5.

The method of Baker and Davenport (see [1]) is used to take care of the other case (see Section 7). In order to apply this method we have to use once again Baker’s method. This time we are faced with linear forms in three logarithms. This linear forms will be estimated from below by a theorem due to Matveev [10].

After the application of the methods of Baker, Davenport and Mignotte we are left to solve 1000 Thue inequalities. This is done by PARI. For details see Section 8.

2 Notations and Thue Equations

Let us prove first that we may assume \( a \geq 0 \).

**Lemma 1** Let \( \alpha(a) \) denote a zero of \( f_a(x) \) and put \( K(a) = \mathbb{Q}(\alpha(a)) \). Then

\[
N_{K(a)/\mathbb{Q}}(x_0 + x_1\alpha(a) + x_2\alpha(a)^2) = m
\]

holds if and only if

\[
N_{K(-a-1)/\mathbb{Q}}(-x_2 - x_1\alpha(-a-1) - x_0\alpha(-a-1)^2) = -m.
\]

In particular each solution to (4) for \( a \) yields a solution for \(-a-1\).
Proof: It is easy to see that $\alpha(a)$ is a root of $f_a(x)$ if and only if $\frac{1}{\alpha(a)}$ is a root of $f_{-a-1}(x)$. As $N_{K(\alpha)/Q}(-\alpha(a)) = -1$ the assertion follows immediately.

Next, we want to transform the norm form inequality (4) into a Thue inequality. Since $x_0, x_1, x_2$ form an arithmetic progression we may write $x_0 = X - Y, x_1 = X$ and $x_2 = X + Y$. Using this notation in (4) we obtain

$$\left| N_{K/Q}(X(1 + \alpha + \alpha^2) - Y(1 - \alpha^2)) \right| \leq |2a + 1|. $$

Expanding the norm on the left side to a polynomial in $X$ and $Y$ we obtain the Thue inequality

$$\left| (a^2 + a + 7)X^3 - (a^2 + a + 7)Y^2 - (2a + 1)Y^3 \right| \leq |2a + 1|. $$

Since we have the restrictions $x_0 < x_1 < x_2$ and $(x_0, x_1, x_2)$ is primitive, we are only interested in solutions with $Y \geq 1$ and $(X, Y)$ is primitive.

For the rest of this paper we will use the following notations: We denote by $f_a \in \mathbb{Z}[X]$ the Thomas polynomial, which is defined as follows:

$$f_a(X) := X^3 - (a - 1)X^2 - (a + 2)X - 1. $$

Let $\alpha := \alpha_1 > \alpha_3 > \alpha_2$ be the three distinct real roots of $f_a$. Furthermore we define $\gamma := 1 + \alpha + \alpha^2, \delta := 1 - \alpha^2$ and $\epsilon := \delta/\gamma$ and denote by $\gamma_1 := \gamma, \gamma_2, \gamma_3, \delta_1 := \delta, \delta_2, \delta_3$ and $\epsilon_1 := \epsilon, \epsilon_2, \epsilon_3$ their conjugates respectively. Moreover we define $G_a \in \mathbb{Z}[X, Y]$ and $g_a \in \mathbb{Z}[X]$ by

$$G_a(X, Y) := (a^2 + a + 7)X^3 - (a^2 + a + 7)XY^2 - (2a + 1)Y^3, $$

$$g_a(X) := G_a(X, 1) = (a^2 + a + 7)X^3 - (a^2 + a + 7)X - (2a + 1). $$

Let us remark that $\epsilon_1, \epsilon_2$ and $\epsilon_3$ are exactly the roots of $g_a$.

If $(X, Y)$ is a solution to (5) then we define $\beta := X\gamma - Y\delta$ and we denote by $\beta_1 := \beta, \beta_2, \beta_3$ the conjugates of $\beta$. As one can easily see $\beta_i$ is an element of the order $\mathbb{Z}[\alpha_i]$ for all $i = 1, \ldots, 3$. In fact the orders $\mathbb{Z}[\alpha_i]$ are all the same (see [15, 17, 18] or Section 4).

There are a lot of well known facts about the number fields $K := \mathbb{Q}(\alpha)$, which we will state in Section 4.

We will use the following variant of the usual $O$-notation: For two functions $g(t)$ and $h(t)$ and a positive number $t_0$ we will write $g(t) = L_{t_0}(h(t))$ if $|g(t)| \leq h(t)$ for all $t$ with absolute value at least $t_0$. We will use this notation in the middle of an expression in the same way as it is usually done with the $O$-notation. Sometimes we omit the index $t_0$. This will happen only in theoretical results, and it means that there exists a (computable) $t_0$ with the desired property.
This $L$-notation will help us to state asymptotic results in a comfortable way.

3 Asymptotic expansions

Due to Thomas [17] we know that

$$\alpha_1 \sim a, \quad \alpha_2 \sim -1, \quad \alpha_3 \sim -1/a.$$  

We apply Newton’s method to the polynomial $f_a$ with starting points $a, -1$ and 0. After 4 steps of Newton’s method and an asymptotic expansion of the resulting expressions we get

$$\tilde{\alpha}_1 := a + \frac{2}{a} - \frac{1}{a^2} - \frac{3}{a^3} + \frac{5}{a^4} \sim \alpha_1,$$

$$\tilde{\alpha}_2 := -1 - \frac{1}{a} + \frac{2}{a^3} - \frac{1}{a^4} \sim \alpha_2,$$

$$\tilde{\alpha}_3 := -\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} - \frac{4}{a^4} \sim \alpha_3.$$  \hspace{1cm} (8)

We consider the quantities $-f_a(\tilde{\alpha}_i + e_i/a^5)f_a(\tilde{\alpha}_i - e_i/a^5)$ with $e_1 = 10, e_2 = 8$ and $e_3 = 18$. These quantities are all positive provided that $a \geq 8, a \geq 7$ and $a \geq 10$ respectively, hence

$$\alpha_1 = a + \frac{2}{a} - \frac{1}{a^2} - \frac{3}{a^3} + \frac{5}{a^4} + L_{1000} \left(\frac{36.037}{a^5}\right),$$

$$\alpha_2 = -1 - \frac{1}{a} + \frac{2}{a^3} - \frac{1}{a^4} + L_7 \left(\frac{8}{a^5}\right),$$

$$\alpha_3 = -\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} - \frac{4}{a^4} + L_{10} \left(\frac{18}{a^5}\right).$$  \hspace{1cm} (9)

Since $\alpha_1 + \alpha_2 + \alpha_3 = a - 1$ is an integer we also obtain

$$\alpha_3 = -\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} - \frac{4}{a^4} + L_8 \left(\frac{18}{a^5}\right).$$

In order to keep the error terms low from now on we assume that $a \geq 1000$. Using these asymptotic expansions we obtain for the $\gamma$’s

$$\gamma_1 = a^2 + a + 5 - \frac{3}{a^2} - \frac{3}{a^3} + L_{1000} \left(\frac{36.037}{a^4}\right),$$

$$\gamma_2 = 1 + \frac{1}{a} + \frac{1}{a^2} - \frac{2}{a^3} - \frac{3}{a^4} + L_{1000} \left(\frac{26.021}{a^5}\right),$$

$$\gamma_3 = 1 - \frac{1}{a} + \frac{2}{a^2} - \frac{1}{a^3} - \frac{5}{a^4} + L_{1000} \left(\frac{28.044}{a^5}\right).$$  \hspace{1cm} (10)
and similarly for the $\delta$'s

$$
\delta_1 = -a^2 - 3 + \frac{2}{t} + \frac{2}{a^2} - \frac{6}{a^3} + L_{1000} \left( \frac{31.027}{a^4} \right),
$$

$$
\delta_2 = -\frac{2}{a} - \frac{1}{a^2} + \frac{4}{a^3} + \frac{2}{a^4} + L_{1000} \left( \frac{18.021}{a^5} \right),
$$

$$
\delta_3 = 1 - \frac{1}{a^2} + \frac{2}{a^3} + \frac{10}{a^4} + L_{1000} \left( \frac{43.045}{a^6} \right),
$$

and for the $\epsilon$'s

$$
\epsilon_1 = -1 + \frac{1}{a} + \frac{1}{a^2} - \frac{4}{a^3} - \frac{2}{a^4} + \frac{2}{a^5} + L_{1000} \left( \frac{108.886}{a^6} \right),
$$

$$
\epsilon_2 = -\frac{2}{a} + \frac{1}{a^2} + \frac{5}{a^3} - \frac{8}{a^4} + L_{1000} \left( \frac{67.81}{a^5} \right),
$$

$$
\epsilon_3 = 1 - \frac{1}{a} - \frac{2}{a^2} - \frac{1}{a^3} + L_{1000} \left( \frac{36.385}{a^4} \right).
$$

We will also use the asymptotic expansions of the logarithms of the $\alpha$'s. Therefore we recall a simple fact from analysis: If $|t| > |r|$ then

$$
\log |t + r| = \log |t| - \sum_{i=1}^{N} \left( \frac{-r/t}{i} \right)^i + L \left( \frac{r}{t} \cdot \frac{N+1}{N+1} \cdot \left| \frac{t}{t-r} \right| \right).
$$

We have omitted the index $t_0$ since this index depends on the $L$-Term of the quantity $r$. Let us write

$$
\alpha = \frac{a^2}{a} + \frac{2}{a} - \frac{1}{a^2} + \frac{3}{a^3} + \frac{5}{a^4} + L_{1000} \left( \frac{10}{a^5} \right).
$$

We can write similar expressions for $\alpha_2$ and $\alpha_3$, too. Using the above formula we get

$$
\log |\alpha_1| = \log a - \frac{2}{a^2} + \frac{1}{a^3} + \frac{5}{a^4} - \frac{7}{a^5} + L_{1000} \left( \frac{18.184}{a^6} \right),
$$

$$
\log |\alpha_2| = -\frac{1}{a} + \frac{1}{2a^2} + \frac{5}{3a^3} - \frac{11}{4a^4} + L_{1000} \left( \frac{11.035}{a^5} \right),
$$

$$
\log |\alpha_3| = -\log a + \frac{1}{a} - \frac{3}{2a^2} + L_{1000} \left( \frac{3.514}{a^3} \right).
$$

4 Auxiliary results

Let us recall first some well known facts about the number field $K = \mathbb{Q}(\alpha)$, where $\alpha$ is a root of the Thomas polynomial $f_\alpha$ (these results can be found in [9, 15, 17, 18]).
Lemma 2 Let \( \alpha \) be a root of the polynomial \( f_a \). Then we have the following facts:

(1) The polynomials \( f_a \) are irreducible for all \( a \in \mathbb{Z} \). Moreover all roots of \( f_a \) are real.
(2) The number fields \( K = \mathbb{Q}(\alpha) \) are cyclic Galois extensions of degree three of \( \mathbb{Q} \) for all \( a \in \mathbb{Z} \).
(3) The roots of \( f_a \) are permuted by the map \( \alpha \mapsto -1 - \frac{1}{\alpha} \).
(4) Any two of \( \alpha_1, \alpha_2, \alpha_3 \) form a fundamental system of units of the order \( \mathbb{Z}[\alpha] \), where \( \alpha_1, \alpha_2, \alpha_3 \) denote the conjugates of \( \alpha \).
(5) Let \( a \geq 0 \). If \( |N_{K/\mathbb{Q}}(\gamma)| \leq 2a + 1 \) then \( \gamma \) is either associated to a rational integer or associated to a conjugate of \( \alpha - 1 \).

Proof: Proofs of these statements can be found in [15, 17, 18, 9] except statement (5) in the case of \( a = 0 \) and \( a = 1 \). The case \( a = 0 \) is trivial. So let us consider the case \( a = 1 \).

If \( \gamma \) fulfills \( |N_{K/\mathbb{Q}}(\gamma)| \leq 3 \) and if \( \gamma \) is not a unit of \( \mathbb{Z}[\alpha] \) then \( (\gamma)|2 \) or \( (\gamma)|(3) \). According to [7, Chapter I, Proposition 25] we have \( (3) = p_1^3 \) with \( p_1 = (\alpha_1 - 1) + (3) = (\alpha_1 - 1) \) and \( (2) = p_2 \), where \( p_1 \) and \( p_2 \) are prime ideals. Therefore \( \gamma \) is a multiple of \( \alpha_1 - 1 \) or 2. Computing the norms yields that \( \gamma \) is associated to \( \alpha_1 - 1 \) or is 0. Therefore we have proved the statement for \( a = 1 \).

Part (5) of Lemma 2 shows that we only have to consider algebraic integers, that are associated to a rational integer or associated to a conjugate of \( \alpha - 1 \). Let us exclude the case that \( \gamma = n\epsilon \) with \( n \neq \pm 1 \in \mathbb{Z} \) and \( \epsilon \in \mathbb{Z}[\alpha]^* \) and \( \gamma \) yields a solution to (4). Since \( \gamma = x_0 + x_1\alpha + x_2\alpha^2 \) with unique \( x_0, x_1, x_2 \in \mathbb{Z} \), also \( \epsilon = \frac{x_0}{n} + \frac{x_1}{n}\alpha + \frac{x_2}{n}\alpha^2 \) yields a solution to (4). Therefore \( n|x_0, x_1, x_2 \). However, \( (x_0, x_1, x_2) \) is primitive, thus \( \gamma \) cannot be associated to a rational integer \( \neq \pm 1 \).

We have to solve the Diophantine inequality (5), therefore we start to exclude all small values of \( Y \).

Lemma 3 Let \( (X,Y) \) be a solution to (5) such that \( Y = 1 \), then \( (X,Y) \) only yields solutions stated in Theorem 1.

Proof: We insert \( Y = 1 \) into (5) and obtain

\[
|((a^2 + a + 7)(X^2 - 1))X - (2a + 1)| \leq 2a + 1.
\]

If we assume \( X \geq 2 \), respectively \( X \leq -2 \), then

\[
6(a^2 + a + 7) - (2a + 1) \leq |((a^2 + a + 7)(X^2 - 1))X - (2a + 1)| \leq 2a + 1
\]
yields a contradiction. Therefore \(|X| \leq 1\) and we only obtain solutions stated in Theorem 1.

Now we investigate approximation properties of solutions \((X, Y)\) to (5). We distinguish three types of solutions. We say that \((X, Y)\) is of type \(j\), if

\[
\left| \frac{X}{Y} - \epsilon_j \right| = \min_{i=1,2,3} \left( \left| \frac{X}{Y} - \epsilon_i \right| \right).
\]

A specific case \(j\) will be called by its roman number. Let us assume that \((X, Y)\) is a solution of type \(j\). Then we have (remember \(\beta_i = X\gamma_i - Y\delta_i\))

\[
2 \left| \frac{\beta_i}{\gamma_i} \right| \geq \left| \frac{\beta_j}{\gamma_j} \right| = \left| X - Y\epsilon_i \right| + \left| X - Y\epsilon_j \right| \geq |Y||\epsilon_i - \epsilon_j|.
\]

Since \(|\beta_1\beta_2\beta_3| \leq 2a + 1\) by the above inequality we obtain

\[
|\beta_j| \leq \frac{2a + 1}{\prod_{i \neq j} |\beta_i|} \leq \frac{8a + 4}{|Y|^2 \prod_{i \neq j} |\gamma_i||\epsilon_j - \epsilon_i|}
\]

or equivalently

\[
\left| \frac{\beta_j}{\gamma_j} \right| \leq \frac{8a + 4}{|Y|^2 |N| K/Q |\gamma| \prod_{i \neq j} |\epsilon_j - \epsilon_i|} =: \frac{c_1}{|Y|^3}
\]

and we also get

\[
\text{sign}(y)\epsilon_j - \frac{c_1}{|Y|^3} \leq \frac{X}{|Y|} \leq \text{sign}(y)\epsilon_j + \frac{c_1}{|Y|^3},
\]

hence

\[
\left| \frac{\beta_j}{\gamma_j} \right| = |Y||\epsilon_j - \epsilon_i| + L \left( \frac{c_1}{Y_0^2} \right) = |Y| \left( |\epsilon_j - \epsilon_i| + L \left( \frac{c_1}{Y_0^2} \right) \right),
\]

where \(Y_0\) is some lower bound for \(|Y|\). Because of Lemma 3 we may assume \(Y_0 \geq 2\). Using the asymptotic expansions (9), (10), (11) and (12) we find

- \(c_1 = \frac{4}{a} + L_{1000} \left( \frac{10.011}{a^2} \right)\) if \(j = 1\);
- \(c_1 = \frac{8}{a} + L_{1000} \left( \frac{4.044}{a^2} \right)\) if \(j = 2\);
- \(c_1 = \frac{4}{a} + L_{1000} \left( \frac{14.633}{a^2} \right)\) if \(j = 3\);

Now we can prove a new lower bound \(Y_0\) for \(|Y|\).

**Lemma 4** If \(a \geq 1000\) and \((X, Y)\) is a primitive solution to (5) such that \(Y > 1\) then \(Y \geq \frac{a}{3.01}\).
Proof: We have to distinguish between three cases $j = 1$, $j = 2$ and $j = 3$. We find from (14) and (12):

\[
\begin{align*}
|X - Y \left(-1 + L_{1000} \left(\frac{1.002}{a}\right)\right)| &\leq \frac{4.011}{Y^2 a}, \\
|X - Y L_{1000} \left(\frac{2.002}{a}\right)| &\leq \frac{8.005}{Y^2 a}, \\
|X - Y \left(1 + L_{1000} \left(\frac{1.003}{a}\right)\right)| &\leq \frac{4.015}{Y^2 a}.
\end{align*}
\]

Some straightforward calculations yield

\[
\begin{align*}
|X + Y| &\leq \frac{4.011}{Y^2 a} + \frac{Y1.002}{a} < \frac{1.51Y}{a}, \\
|X| &\leq \frac{8.005}{Y^2 a} + \frac{Y2.002}{a} < \frac{3.01Y}{a}, \\
|X - Y| &\leq \frac{4.015}{Y^2 a} + \frac{Y1.003}{a} < \frac{1.51Y}{a}.
\end{align*}
\]

We conclude that $X + Y = 0$, $X - Y = 0$ or $X = 0$ if $Y < \frac{a}{3.01}$. But if $X + Y = 0$, $X - Y = 0$ or $X = 0$ we get a contradiction, hence $Y \geq \frac{a}{3.01}$.

\[\blacksquare\]

Let $\sigma$ be the automorphism of $K = \mathbb{Q}(\alpha)$ that is induced by $\alpha \mapsto -1 - \frac{1}{\alpha}$. Then we have $\alpha_i = \sigma^{i-1}\alpha$. From part (5) of Lemma 2 we know that $\beta$ is either a unit, associated to a rational integer or associated to a conjugate of $\alpha_1 - 1$. By the discussion after Lemma 2 we know that $\beta$ is not associated to a rational integer $\neq 1$. Furthermore $\alpha_1$ and $\alpha_2$ form a fundamental system of units of the relevant order $\mathbb{Z}[\alpha]$, hence the linear system

\[
\log |\beta| = b_1 \log |\sigma^{i-1}\alpha_1| + b_2 \log |\sigma^{i-1}\alpha_2| + \log |\sigma^{i-1}\mu| \quad i \neq j (16)
\]

with $\mu$ associated to one of $1, \alpha_1 - 1, \alpha_2 - 1$ or $\alpha_3 - 1$, has a unique integral solution $(b_1, b_2)$. Solving (16) by Cramer’s rule we find

\[
B := \max\{|b_1|, |b_2|\} \leq 2\max_{i \neq j} |\beta| \log |\sigma^{i-1}\mu| \max_{i=1,2,3} |\log |\alpha_i|| \\
\quad := \max_{i \neq j} \left|\log |\beta| - \log |\sigma^{i-1}\mu|\right| c_2 \\
\quad \leq \log |Y| c_2 \left(1 + \frac{\max_{i \neq j} |\gamma_i|}{|\sigma^{i-1}\mu|} \left|\left|\epsilon_j - \epsilon_i + \frac{\gamma_i}{Y_0}\right|\right| \right) \log Y_0 \\
\quad := \log |Y| c_3
\]

We will compute the quantity $c_3$ in Section 5, when we have a better lower bound $Y_0 \leq Y$. 

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Now we will investigate Siegel’s identity. Therefore choose \( i, k \in \{1, 2, 3\} \) such that \( i, j, k \) are all pairwise distinct. We consider the quantity
\[
\frac{\beta_i}{\gamma_i} (\epsilon_j - \epsilon_k) + \frac{\beta_j}{\gamma_j} (\epsilon_k - \epsilon_i) + \frac{\beta_k}{\gamma_k} (\epsilon_i - \epsilon_j) = 0.
\]

Taking into account (14) and (15) we find after some manipulations that
\[
\left| \frac{\beta_i}{\gamma_i} \cdot \frac{\epsilon_k - \epsilon_i}{\epsilon_k - \epsilon_j} \right| = 1 - \frac{\beta_k}{\gamma_k} \cdot \frac{\epsilon_j - \epsilon_i}{\epsilon_j - \epsilon_k} \leq c_1 \frac{1}{|Y|^2} \left( \frac{\epsilon_k - \epsilon_i}{\epsilon_k - \epsilon_j} \right) \cdot \frac{1}{|Y|} \left( \frac{\epsilon_j - \epsilon_i}{|\epsilon_j - \epsilon_i| - \frac{\epsilon_j - \epsilon_i}{\gamma_j^2}} \right) := c_4|Y|^3.
\]

By the asymptotic expansions (9), (10), (11) and (12) together with the bounds for \( c_1 \) and Lemma 4, we see that for any choice of \( i, j, k \) except \( (i, j, k) = (3, 2, 1) \) we have \( c_4 \leq 4.035a \) provided that \( a \geq 1000 \). In the exceptional case we get \( c_4 \leq 4.055a \). Note that this exceptional case, will not occur in this paper.

5 A first bound for the parameter

In this section we will derive a first upper bound for \( a \) such that (5) has no primitive solution \((X, Y)\) with \( Y > 1 \). First we consider
\[
\Lambda_{i,j,k} := \log \left| \frac{\beta_k}{\gamma_k} \cdot \frac{\epsilon_j - \epsilon_i}{\epsilon_j - \epsilon_k} \right| = \log \left( \frac{\gamma_i}{\gamma_k} \cdot \frac{\epsilon_j - \epsilon_i}{\epsilon_j - \epsilon_k} \right) + b_1 \log \left( \frac{\sigma^{k-1} \alpha}{\sigma^{i-1} \alpha} \right) + b_2 \log \left( \frac{\sigma \alpha}{\sigma^{i-1} \mu} \right) + \log \left( \frac{\sigma^{k-1} \mu}{\sigma^{i-1} \mu} \right).
\]

From Siegel’s identity (18) and the fact that \( \log |x| < 2|1 - x| \) provided that \( |1 - x| < 1/3 \) we obtain
\[
|\Lambda_{i,j,k}| < 2 \left| 1 - \frac{\beta_k}{\gamma_k} \cdot \frac{\epsilon_j - \epsilon_i}{\epsilon_j - \epsilon_k} \right| \leq \frac{2c_4}{|Y|^3}.
\]

Let \( \theta_{i,j,k} := \frac{\gamma_i}{\gamma_k} \cdot \frac{\epsilon_j - \epsilon_i}{\epsilon_j - \epsilon_k} \). We want to write \( \Lambda_{i,j,k} \) as a linear combination of the logarithms of \( \theta_{i,j,k}, \sigma^{k-1} \alpha, \sigma^{i-1} \alpha_1 \) and \( \alpha_2 \). Therefore we have to distinguish between
several cases. In particular, we consider the three linear forms:

\[
\begin{align*}
\Lambda_1 := B_1 \log |\alpha_1| + B_2 \log |\alpha_2| + \log \left| \frac{\sigma_{1,2} \mu}{\sigma_{2}^2 \mu} \right| & \quad (i, j, k) = (3, 1, 2), \\
\Lambda_2 := B_1 \log |\alpha_1| + B_2 \log |\alpha_2| + \log \left| \frac{\sigma_{1,2} \mu}{\mu} \right| & \quad (i, j, k) = (1, 2, 3), \\
\Lambda_3 := B_1 \log |\alpha_1| + B_2 \log |\alpha_2| + \log \left| \frac{\sigma_{1,2} \mu}{\sigma_{2}^2 \mu} \right| & \quad (i, j, k) = (1, 3, 2),
\end{align*}
\]

where

\[
\begin{align*}
B_1 & := b_1 - 2b_2 & B_2 & := 2b_1 - b_2 & \text{in case of } \Lambda_1, \\
B_1 & := -2b_1 + b_2 & B_2 & := -b_1 - b_2 & \text{in case of } \Lambda_2, \\
B_1 & := b_1 + b_2 & B_2 & := -b_1 + 2b_2 & \text{in case of } \Lambda_3.
\end{align*}
\]

Let us find relations between \(B_2\) and \(B\). These will be used in view of (17). Below we will distinguish between the case of \(B_1 = 0\) and \(B_1 \neq 0\). Let us consider case I: Since \(B = \max\{|b_1|, |b_2|\}\) we have trivially \(|B_2| \leq 3B\). If we assume \(B_1 = 0\) then we have \(b_1 = 2b_2\) and therefore \(B = |b_1|\). Inserting this relation in the equation for \(B_2\) we get \(B_2 = \frac{3}{2}b_1\), hence \(|B_2| = \frac{3}{2}B\). The two other cases are similar and the relations are given in table 2.

Table 2

<table>
<thead>
<tr>
<th></th>
<th>Case I</th>
<th>Case II</th>
<th>Case III</th>
</tr>
</thead>
<tbody>
<tr>
<td>(B_1 \neq 0)</td>
<td>(</td>
<td>B_2</td>
<td>\leq 3B</td>
</tr>
<tr>
<td>(B_1 = 0)</td>
<td>(</td>
<td>B_2</td>
<td>= \frac{3}{2}B</td>
</tr>
</tbody>
</table>

We have to distinguish between 12 cases (three linear forms and for each linear form four possible choices for \(\mu\)). Since all 12 cases can be treated similarly, we only consider the case of \(\Lambda_1\) and \(\mu\) being associated to \(\alpha_2 - 1\). We choose this case because it is representative for most of the other cases. The computed quantities for the other cases are presented in tables. To say that \(\mu\) is associated to some quantity \(\alpha\) we use the notation \(\mu \sim \alpha\).

By (19) and (20) we find

\[
|\Lambda_1| = B_1 \left( \log a - \frac{2}{a^2} + \frac{1}{a^3} + \frac{5}{a^4} - \frac{7}{a^5} + L \left( \frac{18.13370123}{a^6} \right) \right) \\
+ B_2 \left( -\frac{1}{a} + \frac{1}{2a^2} + \frac{5}{3a^3} - \frac{11}{4a^4} + L \left( \frac{11.035}{a^5} \right) \right) + \log \left| \frac{\sigma_{1,2} \mu}{\sigma_{2}^2 \mu} \right|
\]

\[
\leq c_4 \frac{220.1}{a^2}.
\]

By this inequality we see that \(B_2\) has to be large with respect to \(B_1\), except the main terms of \(B_1 \log |\alpha_1|\) and \(\log |\theta_{1,2} \sigma_{2} / \sigma_{2}^2 \mu|\) cancel. We want to choose
\(\mu\) such that a cancelation may only occur if \(B_1 = 0\). Since \(\theta_{3,1,2} = \log 2 + \cdots\) we have to choose \(\mu\) such that \(\mu \sim \alpha_2 - 1\) and \(\sigma \mu / \sigma^2 \mu = O(1)\). With this constraints we choose \(\mu = (\alpha_2 - 1) \alpha_1\). The other choices for \(\mu\) are given in table 3.

### Table 3

<table>
<thead>
<tr>
<th>Case</th>
<th>(\mu \sim )</th>
<th>(\mu \sim \alpha_1 - 1)</th>
<th>(\mu \sim \alpha_2 - 1)</th>
<th>(\mu \sim \alpha_3 - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>1</td>
<td>(\alpha_1 - 1)</td>
<td>((\alpha_2 - 1) \alpha_1)</td>
<td>(\frac{\alpha_3 - 1}{\alpha_1})</td>
</tr>
<tr>
<td>Case II</td>
<td>(\alpha_1)</td>
<td>(\frac{\alpha_1 - 1}{\alpha_2})</td>
<td>(\frac{(\alpha_2 - 1) \alpha_1}{\alpha_3})</td>
<td>(\frac{(\alpha_3 - 1) \alpha_2}{\alpha_3})</td>
</tr>
<tr>
<td>Case III</td>
<td>(\frac{1}{\alpha_3})</td>
<td>((\alpha_1 - 1) \alpha_1)</td>
<td>(\frac{\alpha_2 - 1}{\alpha_3})</td>
<td>(\frac{(\alpha_3 - 1) \alpha_2}{\alpha_3})</td>
</tr>
</tbody>
</table>

Now we distinguish between two further cases: \(B_1 = 0\) and \(B_1 \neq 0\). In the case of \(B_1 = 0\) we have

\[|\Lambda_1| = B_2 \left( \frac{-1}{a} + \frac{1}{2a^2} + \frac{5}{3a^3} - \frac{11}{4a^4} + L \left( \frac{11.035}{a^5} \right) \right) + \log 2 - \frac{5}{a} + \frac{2}{a^2} + L \left( \frac{162.8341694}{a^5} \right) = L \left( \frac{220.1}{a^2} \right).\]

Solving this equation for \(B_2\), we obtain

\[B_2 = a \log 2 + \frac{\log 2}{2} - 5 + L \left( \frac{233.7804338}{a} \right).\] (23)

In the case of \(B_1 \neq 0\) we similarly determine the quantity

\[\frac{B_2}{B_1} = a \log a + \frac{\log a}{2} + \frac{a \log 2 + \log 2}{B_1} - 5 + L \left( \frac{46.920379 \cdot \log a}{a} \right).\] (24)

The results obtained in the other cases are listed in table 4.

Looking at table 4 we see that in the case of \(B_1 = 0\) two different phenomena occur. In the cases I (\(\mu \sim \alpha_3 - 1\)), II (\(\mu \sim 1\)), \(\Pi (\mu \sim \alpha_1 - 1, \Pi (\mu \sim \alpha_2 - 1)\) and III (\(\mu \sim \alpha_2 - 1\)) the quantity \(B_2\) is of the form constant plus some error term, while in the other cases \(B_2\) is constant times \(\log a\) plus lower terms. We are interested in the former cases. In case I (\(\mu \sim \alpha_3 - 1\)), II (\(\mu \sim \alpha_2 - 1\)) and III (\(\mu \sim \alpha_2 - 1\)) \(B_2\) cannot be an integer if \(a \geq 500\). However, by definition \(B_2\) is an integer, so we have a contradiction. In the cases of II (\(\mu \sim 1\)) respectively II (\(\mu \sim \alpha_1 - 1\)) we have \(B_2 = 1\) respectively \(B_2 = 5\) provided \(a \geq 500\). Therefore we have the following two linear systems:

\[-2b_1 + b_2 = 0,\]
\[-b_1 - b_2 = 1,\]

and

\[-2b_1 + b_2 = 0,\]
\[-b_1 - b_2 = 5.\]
Table 4
The quantities $B_2$ and $B_2/B_1$.

| Case I | $\mu \sim 1$ | $B_2 = a \log 2 + \frac{\log 2}{2} - 1 + L \left( \frac{233.5726034}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{a \log 2 + \log 2 - 1}{B_1} + L \left( \frac{46.89229255-\log a}{a} \right)$ |
|        | $\mu \sim \alpha_1 - 1$ | $B_2 = a \log 4 + \log 2 - \frac{1}{2} + L \left( \frac{243.5541701}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{a \log 4 + \log 2 - \frac{1}{2}}{B_1} + L \left( \frac{48.33527238-\log a}{a} \right)$ |
|        | $\mu \sim \alpha_2 - 1$ | $B_2 = a \log 2 + \frac{\log 2}{2} - 5 + L \left( \frac{233.7804338}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{a \log 2 + \log 2 - 5}{B_1} + L \left( \frac{46.920379-\log a}{a} \right)$ |
|        | $\mu \sim \alpha_3 - 1$ | $B_2 = -\frac{1}{2} + L \left( \frac{233.5738003}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} - \frac{1}{2B_1} + L \left( \frac{45.44346894-\log a}{a} \right)$ |

| Case II | $\mu \sim 1$ | $B_2 = 5 + L \left( \frac{225.5761744}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{5}{B_1} + L \left( \frac{45.7326909-\log a}{a} \right)$ |
|        | $\mu \sim \alpha_1 - 1$ | $B_2 = 1 + L \left( \frac{221.7360355}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{1}{B_1} + L \left( \frac{45.17677378-\log a}{a} \right)$ |
|        | $\mu \sim \alpha_2 - 1$ | $B_2 = \frac{15}{2} + L \left( \frac{231.7758252}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{15}{2B_1} + L \left( \frac{46.63018224-\log a}{a} \right)$ |
|        | $\mu \sim \alpha_3 - 1$ | $B_2 = a \log 4 + \log 2 + \frac{9}{2} + L \left( \frac{248.3704756}{a} \right)$ |
|        | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{a \log 4 + \log 2 + \frac{9}{2}}{B_1} + L \left( \frac{49.03250394-\log a}{a} \right)$ |

| Case III | $\mu \sim 1$ | $B_2 = a \log 2 + \frac{\log 2}{2} + \frac{4}{a} + L \left( \frac{233.8513408}{a} \right)$ |
|          | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{a \log 2 + \log 2 + \frac{4}{a}}{B_1} + L \left( \frac{47.50970317-\log a}{a} \right)$ |
|          | $\mu \sim \alpha_1 - 1$ | $B_2 = a \log 4 + \log 2 + \frac{1}{2} + L \left( \frac{244.3001410}{a} \right)$ |
|          | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{a \log 4 + \log 2 + \frac{1}{2}}{B_1} + L \left( \frac{48.4432664-\log a}{a} \right)$ |
|          | $\mu \sim \alpha_2 - 1$ | $B_2 = \frac{7}{2} + L \left( \frac{227.3839598}{a} \right)$ |
|          | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{7}{2B_1} + L \left( \frac{45.99439458-\log a}{a} \right)$ |
|          | $\mu \sim \alpha_3 - 1$ | $B_2 = a \log 2 + \frac{\log 2}{2} + \frac{8}{2} + L \left( \frac{242.3186056}{a} \right)$ |
|          | $B_1$ | $= a \log a + \frac{\log a}{2} + \frac{a \log 2 + \log 2 + \frac{8}{2}}{B_1} + L \left( \frac{48.15640604-\log a}{a} \right)$ |

Solving these systems we find $b_1 = -1/3, b_2 = -2/3$ and $b_1 = -5/3, b_2 = -10/3$. By definition $b_1$ and $b_2$ have to be integers, hence we have again a contradiction. Therefore we may exclude the cases I ($\mu \sim \alpha_3 - 1$), II ($\mu \sim 1$), II ($\mu \sim \alpha_1 - 1$), II ($\mu \sim \alpha_2 - 1$) and III ($\mu \sim \alpha_2 - 1$), if we assume $B_1 = 0$.

Next, we want to estimate the quantity $c_3$ and find a lower bound for $\log Y$. 

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From (23) and (24) we find

\[ B_2 = a \log 2 + \frac{\log 2}{2} - 5 + L \left( \frac{233.781}{a} \right) \geq 0.6883a \quad (25) \]

\[ |B_2| = |B_1| \left( a \log a + \frac{\log a}{2} \right) + a \log 2 + \frac{\log 2}{2} - 5 + L \left( \frac{46.921 \cdot \log a}{a} \right) \geq 6.223a, \quad (26) \]

respectively. Let us estimate the quantity \( c_2 \). From (17) and (13) we find \( c_2 \leq 2.0006 \log a \). Now we are ready to estimate the quantity \( c_3 \). Put

\[ \tilde{c} := 1 + \left| \log \max_{i \neq j} \frac{|\gamma_i|}{|\alpha_i^*|} \left( |\epsilon_j - \epsilon_i| + \frac{c_1}{|Y_0|} \right) \right| \log |Y_0|. \]

Using Lemma 3 together with the asymptotic expansions from Section 3 we obtain

\[ \tilde{c} \leq 1 + \frac{0.5826}{\log a} - \frac{0.8405}{a \log a} + L \left( \frac{52.376}{a^3 \log a} \right) \]

and from the bound for \( c_2 \) we find

\[ c_3 \leq \frac{2.006}{\log a} + \frac{1.1655}{(\log a)^2} - \frac{1.682}{a(\log a)^2} + L \left( \frac{104.782}{a^3(\log a)^2} \right) \leq \frac{2.169079894}{\log a}. \]

Since we have lower bounds for \( B_2 \), hence also for \( B \), and upper bounds for \( c_3 \), using table 2 and inequality (17) we find that:

\[ \log Y \geq 1.4612a \text{ if } B_1 = 0, \]
\[ \log Y \geq 6.6053a \text{ if } B_1 \neq 0. \]

Computing again \( c_3 \) using this time instead of Lemma 3 the new bounds found for \( \log Y \) we get “better” results. Iterating this procedure four times yields:

\[ c_3 \leq \frac{2.00148}{\log a} \text{ and } \log Y \geq 1.5836a \text{ if } B_1 = 0, \text{ respectively} \]
\[ c_3 \leq \frac{2.0008}{\log a} \text{ and } \log Y \geq 7.1609a \text{ if } B_1 \neq 0. \]

The bounds for \( c_3 \) and \( \log Y \) that are obtained in the other cases are listed in table 5 and table 6.

In the next step we use a powerful theorem on lower bounds for linear forms in two logarithms due to Laurent, Mignotte, and Nesterenko [8].

**Lemma 5** Let \( \alpha_1 \) and \( \alpha_2 \) be two multiplicatively independent elements in a number field of degree \( D \) over \( \mathbb{Q} \). For \( i = 1 \) and \( i = 2 \), let \( \log \alpha_i \) be any determination of the logarithm of \( \alpha_i \), and let \( A_i > 1 \) be a real number satisfying

\[ \log A_i \geq \max\{ h(\alpha_i), |\log \alpha_i|/D, 1/D \}, \]

\[ \log A_i \geq \max\{ h(\alpha_i), |\log \alpha_i|/D, 1/D \}, \]

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Table 5
Upper bounds for $c_3$.

<table>
<thead>
<tr>
<th>$c_3 \leq$</th>
<th>$\mu \sim 1$</th>
<th>$\mu \sim \alpha_1 - 1$</th>
<th>$\mu \sim \alpha_2 - 1$</th>
<th>$\mu \sim \alpha_3 - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>2.001471859/\log a</td>
<td>2.001035919/\log a</td>
<td>2.00147401/\log a</td>
<td>/</td>
</tr>
<tr>
<td>$B_1 = 0$</td>
<td>2.000794053/\log a</td>
<td>2.000818748/\log a</td>
<td>2.000793370/\log a</td>
<td>2.002338185/\log a</td>
</tr>
<tr>
<td>$B_1 \neq 0$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>2.00926921/\log a</td>
</tr>
<tr>
<td>Case II</td>
<td>2.01760135/\log a</td>
<td>2.001759126/\log a</td>
<td>2.000705217/\log a</td>
<td>2.001890017/\log a</td>
</tr>
<tr>
<td>$B_1 = 0$</td>
<td>2.019731368/\log a</td>
<td>2.001472020/\log a</td>
<td>/</td>
<td>2.019611578/\log a</td>
</tr>
<tr>
<td>$B_1 \neq 0$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>Case III</td>
<td>2.002728648/\log a</td>
<td>2.000818944/\log a</td>
<td>2.002339951/\log a</td>
<td>2.002730579/\log a</td>
</tr>
<tr>
<td>$B_1 = 0$</td>
<td>2.019731368/\log a</td>
<td>2.001472020/\log a</td>
<td>/</td>
<td>2.019611578/\log a</td>
</tr>
<tr>
<td>$B_1 \neq 0$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
</tr>
</tbody>
</table>

Table 6
Lower bounds for $\log Y$.

<table>
<thead>
<tr>
<th>$\log Y \geq$</th>
<th>$\mu \sim 1$</th>
<th>$\mu \sim \alpha_1 - 1$</th>
<th>$\mu \sim \alpha_2 - 1$</th>
<th>$\mu \sim \alpha_3 - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>1.5928a</td>
<td>3.1902a</td>
<td>1.5836a</td>
<td>/</td>
</tr>
<tr>
<td>$B_1 = 0$</td>
<td>7.1563a</td>
<td>6.3575a</td>
<td>7.1609a</td>
<td>7.9477a</td>
</tr>
<tr>
<td>$B_1 \neq 0$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>Case II</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>1.6026a</td>
</tr>
<tr>
<td>$B_1 = 0$</td>
<td>11.915a</td>
<td>11.922a</td>
<td>13.112a</td>
<td>10.717a</td>
</tr>
<tr>
<td>$B_1 \neq 0$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
</tr>
<tr>
<td>Case III</td>
<td>0.7949a</td>
<td>1.5959a</td>
<td>/</td>
<td>0.8a</td>
</tr>
<tr>
<td>$B_1 = 0$</td>
<td>7.1436a</td>
<td>6.3564a</td>
<td>7.9431a</td>
<td>7.1390a</td>
</tr>
<tr>
<td>$B_1 \neq 0$</td>
<td>/</td>
<td>/</td>
<td>/</td>
<td>/</td>
</tr>
</tbody>
</table>

where $h(\alpha_i)$ denotes the absolute logarithmic Weil height of $\alpha_i$. Further, let $b_1$ and $b_2$ be two positive integers. Define

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1} \quad \text{and} \quad \log b = \max \left\{ \log b', 21/D, \frac{1}{2} \right\}.$$

Then

$$|b_2 \log \alpha_2 - b_1 \log \alpha_1| \geq \exp \left( -30.9 D^4 (\log b)^2 \log A_1 \log A_2 \right).$$

Before we apply this result we have to compute some heights:

**Lemma 6** Let $h$ denote the absolute logarithmic Weil height, then

$$h(\alpha_1) = h(\alpha_2) = h(\alpha_3) \leq \frac{\log a}{3} \quad (27)$$

and

$$h\left( \theta_{3,1,2} \frac{\sigma \mu}{\sigma^2 \mu} \right) \leq \frac{4 \log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.047}{a^3} \quad (a \geq 1000), \quad (28)$$

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where $\mu = (\alpha_2 - 1)\alpha_1$. The estimations for $H := h\left(\theta_{i,j,k} \frac{\sigma^{k-1} \mu}{\sigma^{i-1} \mu}\right)$ in the other cases are given in table 7.

Table 7

<table>
<thead>
<tr>
<th>Case</th>
<th>$\mu \sim 1$</th>
<th>$H \leq \frac{4 \log a}{3} + \frac{\log 2}{3} + \frac{1}{3a} + \frac{5}{3a^2} + \frac{90.0595}{a^3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>$\mu \sim \alpha_1 - 1$</td>
<td>$H \leq \frac{4 \log a}{3} + \frac{\log 8}{3} - \frac{1}{3a} + \frac{9}{4a^2} + \frac{83.3557}{a^3}$</td>
</tr>
<tr>
<td>Case I</td>
<td>$\mu \sim \alpha_2 - 1$</td>
<td>$H \leq \frac{4 \log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.0466}{a^3}$</td>
</tr>
<tr>
<td>Case I</td>
<td>$\mu \sim \alpha_3 - 1$</td>
<td>$H \leq \frac{4 \log a}{3} + \frac{\log 4}{3} - \frac{4}{3a} + \frac{35}{12a^2} + \frac{146.3174}{a^3}$</td>
</tr>
</tbody>
</table>

| Case II  | $\mu \sim 1$ | $H \leq \frac{5 \log a}{3} + \frac{\log 2}{3} + \frac{121}{3a^2} + \frac{187.7049}{a^3}$ |
| Case II  | $\mu \sim \alpha_2 - 1$ | $H \leq \frac{4 \log a}{3} + \frac{7}{6a} + \frac{29}{8a^2} + \frac{301.579}{a^3}$ |
| Case II  | $\mu \sim \alpha_3 - 1$ | $H \leq \frac{5 \log a}{3} + \frac{\log 4}{3} + \frac{17}{24a^2} + \frac{120.6103}{a^3}$ |

| Case III | $\mu \sim 1$ | $H \leq \frac{4 \log a}{3} + \frac{10}{3a^2} + \frac{92.4204}{a^3}$ |
| Case III | $\mu \sim \alpha_2 - 1$ | $H \leq \frac{5 \log a}{3} + \frac{5}{3a} + \frac{35}{12a^2} + \frac{97.6902}{a^3}$ |
| Case III | $\mu \sim \alpha_3 - 1$ | $H \leq \frac{5 \log a}{3} + \frac{\log 4}{3} + \frac{5}{3a} + \frac{25}{12a^2} + \frac{232.4536}{a^3}$ |

**Proof:** We start with the proof of (27). Since $\alpha_1, \alpha_2, \alpha_3$ are conjugate, we only have to check the last inequality.

$$h(\alpha_1) = \frac{1}{3} \left( \sum_{i=1,2,3} \max(0, \log |\alpha_i|) \right) = \frac{1}{3} \left( \log a - \frac{1}{a} - \frac{3}{2a^2} + \frac{8}{3a^3} + L \left( \frac{2.27}{a^4} \right) \right) \leq \frac{\log a}{3},$$

therefore we obtain the first part of the lemma.

Since $\theta_{3,1,2}$ and $\frac{\sigma^\mu}{\sigma^2 \mu}$ are not integers in general we also have to compute their denominators, which can be estimated by

$$\Delta_\theta := N_{K/\mathbb{Q}}(\gamma_1(\epsilon_2 - \epsilon_1)) = a^2 + 2a - 13 \quad \text{respectively,}$$

$$\Delta_\mu := N_{K/\mathbb{Q}}(\alpha_2 - 1) = 2a + 1.$$

With this preliminary result we obtain

$$h\left(\theta_{3,1,2} \frac{\sigma^\mu}{\sigma^2 \mu}\right) \leq \frac{1}{3} \left( \log(\Delta_\theta \Delta_\mu) + \sum_{j=1,2,3} \max \left( 0, \log \left|\sigma^j \left(\theta_{3,1,2} \frac{\sigma^\mu}{\sigma^2 \mu}\right)\right| \right) \right) = \frac{4 \log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + L \left( \frac{190.047}{a^3} \right).$$
Now we apply Lemma 5 to the linear form (20). We distinguish between the case of $B_1 = 0$ and $B_1 \neq 0$. In the case of $B_1 = 0$ we can apply Lemma 5 at once. In the notation of Lemma 5 we have

$$b' = \frac{1}{\log a} + \frac{B_2}{4 \log a + \log 2 - \frac{1}{a} + \frac{9}{a^2} + \frac{570.141}{a^3}} \leq \frac{1}{\log a} + \frac{a \log 2 + \log 2}{4 \log a + \log 2 - \frac{1}{a} + \frac{9}{a^2} + \frac{570.141}{a^3}} \leq \frac{a}{\log a} - 0.16898$$

Inserting the various bounds we obtain

$$\log |\Lambda_1| > -834.3(\log a - \log \log a - 1.778)^2 \log a \times \left(\frac{4 \log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.047}{a^3}\right).$$

On the other hand we have from (19)

$$\log |\Lambda_1| < \log \frac{2 c_4}{Y^3} < \log (8.07a) - 0.99926 \times \left(a \log 2 + \frac{\log 2}{2} - 5 - \frac{233.781}{a}\right) \log a.$$

Comparing the upper and lower bound for $\log |\Lambda_1|$ yields a contradiction for large $a$. In particular, if $a \geq 2529022.366$ we have a contradiction. Since $a$ has to be an integer we know that we may have solutions with $|Y| \geq 2$ only if $a \leq a_0 := 2529022$.

Now we investigate the case $B_1 \neq 0$. In this case we do not have a linear form in two logarithms. But we can study the linear form

$$\Lambda_1 = \log \left(\alpha_1 B_1 \theta_{3, 1, 2} \theta_{2, \sigma, \mu}\right) + B_2 \log \alpha_2.$$ 

Since $h(xy) \leq h(x) + h(y)$ we have $h \left(\alpha_1 B_1 \theta_{3, 1, 2} \theta_{2, \sigma, \mu}\right) \leq |B_1| h(\alpha_1) + h \left(\theta_{3, 1, 2} \theta_{2, \sigma, \mu}\right)$ and because of Lemma 6 we choose

$$b' = \frac{1}{\log a} + \frac{|B_2|}{|B_1| \log a + 4 \log a + \log 2 - \frac{1}{a} + \frac{9}{a^2} + \frac{570.141}{a^3}} \leq \frac{1}{\log a} + \frac{|B_2|}{|B_1| \log a} \leq \frac{1}{\log a} + \frac{a \log a + \frac{\log a}{2} + a \log 2 + \frac{\log 2}{2} - 5 + \frac{46.921 \log a}{a}}{\log a} \leq 1.10037a$$
By Lemma 5 we find

\[
\log |\Lambda_1| > -834.3(\log a + 0.0957)^2 \log a \\
\times \left( \frac{|B_1| \log a}{3} + \frac{4 \log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.05}{a^3} \right) \\
\geq -834.3(\log a + 0.0957)^2 \log a |B_2| \frac{|B_1|}{|B_2|} \\
\times \left( \frac{5 \log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.05}{a^3} \right) \\
> -834.3 \frac{(\log a + 0.0957)^2 \log a |B_2| (\frac{5}{3} \log a + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.05}{a^3})}{a \log a + \frac{\log a}{2} - a \log 2 - \frac{\log 2}{2} + 5 - \frac{46.921 \log a}{a}}
\]

On the other hand

\[
\log |\Lambda_1| < \log 2c_4 - 3 \log Y \leq \log 8.07 + \log a - \frac{3B}{c_3}
\]

\[
\leq |B_2| \left( \frac{\log 8.07 + \log a}{B_2} - \frac{1}{c_3} \right)
\]

\[
\leq |B_2| \left( \frac{\log 8.07 + \log a}{a \log a + \frac{\log a}{2} - a \log 2 - \frac{\log 2}{2} + 5 - \frac{46.921 \log a}{a}} - \frac{\log a}{2.000793370} \right)
\]

If we compare these bounds for \( \log |\Lambda_1| \) we see that \( |B_2| \) cancels, and we obtain an inequality which cannot hold for \( a \geq 521855.0066 \). That is, if there is a solution not found yet for this case, then \( a \leq a_0 := 521855 \).

In table 8 one finds the other upper bounds \( a_0 \) of the parameter \( a \) for the remaining cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>( B_1 = 0 )</th>
<th>( B_1 \neq 0 )</th>
<th>( B_1 = 0 )</th>
<th>( B_1 \neq 0 )</th>
<th>( B_1 = 0 )</th>
<th>( B_1 \neq 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>( a_0 = 2532736 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
</tr>
<tr>
<td>II</td>
<td>( a_0 = 1226494 )</td>
<td>( a_0 = 579982 )</td>
<td>( a_0 = 521855 )</td>
<td>( a_0 = 487789 )</td>
<td>( a_0 = 3259385 )</td>
<td>( a_0 = 3259385 )</td>
</tr>
<tr>
<td></td>
<td>( a_0 = 229399 )</td>
<td>( a_0 = 377086 )</td>
<td>( a_0 = 521855 )</td>
<td>( a_0 = 405414 )</td>
<td>( a_0 = 3059080 )</td>
<td>( a_0 = 3059080 )</td>
</tr>
<tr>
<td>III</td>
<td>( a_0 = 2529022 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
<td>( a_0 = 521904 )</td>
</tr>
<tr>
<td></td>
<td>( a_0 = 397229 )</td>
<td>( a_0 = 579994 )</td>
<td>( a_0 = 590044 )</td>
<td>( a_0 = 651927 )</td>
<td>( a_0 = 8157825 )</td>
<td>( a_0 = 8157825 )</td>
</tr>
</tbody>
</table>

By table 8 we have:

**Proposition 1** There are no other solutions to (4) than those listed in Theorem 1 if \( a > 8157825 \).
The method of Mignotte

In this section we want to eliminate the case of $B_1 = 0$. We have already discussed the cases I ($\mu \sim \alpha_3 - 1$), II ($\mu \sim 1$), II ($\mu \sim \alpha_1 - 1$), II ($\mu \sim \alpha_2 - 1$) and III ($\mu \sim \alpha_2 - 1$). We know that $B_2$ has to be an integer therefore let us compute $B_2$ to a higher asymptotic order (in the remaining cases):

$$B_2 = a \log 2 - \frac{2 - \log 2}{2} - \frac{54 - 23 \log 2}{12a} + L \left( \frac{9.4241}{a^2} + 8.075a e^{-4.7784a} \right)$$

- **case I ($\mu \sim 1$)**

$$B_2 = a \log 4 - \frac{1 - \log 4}{2} - \frac{135 - 46 \log 2}{24a} + L \left( \frac{8.528}{a^2} + 8.075a e^{-9.5706a} \right)$$

- **case I ($\mu \sim \alpha_1 - 1$)**

$$B_2 = a \log 2 - \frac{10 - \log 2}{2} - \frac{54 - 23 \log 2}{12a} + L \left( \frac{11.4221}{a^2} + 8.075a e^{-4.7508a} \right)$$

- **case I ($\mu \sim \alpha_2 - 1$)**

$$B_2 = a \log 2 + \frac{11 + \log 2}{2} - \frac{27 - 46 \log 2}{24a} + L \left( \frac{24.2511}{a^2} + 8.075a e^{-4.8078a} \right)$$

- **case II ($\mu \sim \alpha_3 - 1$)**

$$B_2 = a \log 2 + \frac{8 + \log 2}{2} - \frac{54 - 23 \log 2}{12a} + L \left( \frac{13.9461}{a^2} + 8.075a e^{-2.3847a} \right)$$

- **case III ($\mu \sim 1$)**

$$B_2 = a \log 4 + \frac{1 + \log 4}{2} - \frac{135 - 46 \log 2}{24a} + L \left( \frac{14.1731}{a^2} + 8.075a e^{-4.7877a} \right)$$

- **case III ($\mu \sim \alpha_1 - 1$)**

$$B_2 = a \log 2 + \frac{16 - \log 2}{2} - \frac{54 - 23 \log 2}{12a} + L \left( \frac{15.9481}{a^2} + 8.075a e^{-2.4a} \right)$$

Since $B_2$ has to be an integer, for each case we have a criteria wether there exists a solution such that $B_1 = 0$ for one specific $a$. For example, the case I ($\mu \sim \alpha_2 - 1$) yields following criteria:

**Lemma 7** Let $\| \cdot \|$ denote the distance to the nearest integer. If (4) has a solution, which is not found yet, that corresponds to the case I ($\mu \sim \alpha_2 - 1$) such that $B_1 = 0$, then

$$\left\| a \log 2 - \frac{10 - \log 2}{2} - \frac{54 - 23 \log 2}{12a} \right\| \leq \frac{11.4221}{a^2} + 8.075ae^{-4.7508a}.$$ 

The other cases yield similar criteria. Therefore, in the case of $B_1 = 0$ and
I (µ ∼ 1), I (µ ∼ α_1 - 1), I (µ ∼ α_2 - 1), II (µ ∼ α_3 - 1), III (µ ∼ 1), III (µ ∼ α_1 - 1) or III (µ ∼ α_3 - 1) we check for each 1000 ≤ a ≤ a_0 whether the corresponding criteria is fulfilled or not. A computation in MAGMA (see Section 8) yields:

**Proposition 2** If (X, Y) is a solution to (5) with Y ≥ 1 which yields a solution to (4) that is not listed in Theorem 1, then a ≤ 651957. Moreover the solution (X, Y) yields B_1 ≠ 0 or a < 1000.

**Remark 1** This method is called Mignotte’s method, because Mignotte [11] used a similar trick to solve the family of Thue equations

\[ X^3 - (n - 1)X^2Y - (n + 2)XY^2 - Y^3 = 1 \]

completely.

7 The method of Baker and Davenport

We cannot use the method described above to solve the case of B_1 ≠ 0, because we have found an upper bound for the quantity B_2 but not for B_2 itself, which would be essential. So we are forced to use another method. We choose the method of Baker and Davenport [1]. In particular we adapt a lemma of Mignotte, Pethő and Roth [12] to our needs.

In order to use the method of Baker and Davenport, we have to find an absolute lower bound for B_2. Therefore we have to revise the linear forms Λ_1, Λ_2 and Λ_3. This time we do not consider them as linear combinations of two logarithms but as three logarithms. So we cannot use the theorem of Laurent, Mignotte and Nesterenko [8] and have to apply a result of Matveev [10]:

**Lemma 8** Denote by α_1, ..., α_n algebraic numbers, not 0 or 1, by log α_1, ..., log α_n determinations of their logarithms, by D the degree over \( \mathbb{Q} \) of the number field \( \mathbb{K} = \mathbb{Q}(α_1, ..., α_n) \), and by b_1, ..., b_n rational integers. Furthermore let \( κ = 1 \) if \( \mathbb{K} \) is real and \( κ = 2 \) otherwise. Define

\[
\log A_i = \max\{Dh(α_i), |\log α_i|\} \quad (1 ≤ i ≤ n),
\]

where \( h(α) \) denotes the absolute logarithmic Weil height of α and

\[
B^* = \max\{1, \max\{|b_j|A_j/A_n : 1 ≤ j ≤ n\}\}.
\]

Assume that \( b_n ≠ 0 \) and log α_1, ..., log α_n are linearly independent over \( \mathbb{Z} \); then

\[
\log |Λ| ≥ -C(n)C_0W_0D^2Ω,
\]
\[ \Omega = \log(A_1) \cdots \log(A_n), \]

\[ C(n) = C(n, \kappa) = \frac{16}{n^{1/\kappa}} e^n (2n + 1 + 2\kappa)(n + 2)(4(n + 1))^{n+1} \left( \frac{1}{2} e^n \right)^\kappa, \]

\[ C_0 = \log \left( e^{4.4n + 7} n^{3.5} D^2 \log(eD) \right), \quad W_0 = \log(1.5eB^* D \log(eD)). \]

We already have computed all relevant heights in Lemma 6 respectively table 7. We combine Siegel's identity (18) with Matveev's lower bound (Lemma 8) and obtain for our standard case I \((\mu \sim \alpha_2 - 1)\):

\[
\frac{|B_2| \log a}{2.000793370} - \log 8.07 - \log a < \\
1.691497 \cdot 10^{11} (\log a)^2 \left( \frac{4 \log a}{3} + \frac{\log 2}{3} - \frac{1}{3a} + \frac{3}{a^2} + \frac{190.047}{a^3} \right) \log(2.26688|B_2|). \tag{29}
\]

The only not straightforward step is to compute \(B^*\). Therefore let us rearrange the terms of \(\Lambda_j\) such that the term \(\theta_{i,j,k}^{(\sigma - i)/\mu}\) is the last one. Since in any case \(|B_2| > |B_1|\) and \(|B_2| > a \geq 1000\) we have \(B^* = |B_2| \frac{\log a}{4 \log a + \log 2 + \ldots} \leq \frac{|B_2|}{4}\). The inequality (29) yields a contradiction if \(|B_2|\) is large, i.e. \(|B_2| \geq c_5\), where \(c_5\) is some quantity depending on \(a\). In view of an absolute lower bound for \(|B_2|\) the ”worst” case occurs, if \(a\) is as large as possible. Therefore we insert \(a_0\) instead of \(a\) into the inequality above and by solving this inequality we obtain \(|B_2| > 8.93 \cdot 10^{15}\). The lower bounds for \(|B_2|\) in the other cases can be found in table 9.

**Table 9**

| \(|B_2|\) > | \(\mu \sim 1\) | \(\mu \sim \alpha_1 - 1\) | \(\mu \sim \alpha_2 - 1\) | \(\mu \sim \alpha_3 - 1\) |
|---|---|---|---|---|
| Case I | 8.92 \cdot 10^{15} | 9.31 \cdot 10^{15} | 8.92 \cdot 10^{15} | 8.95 \cdot 10^{15} |
| Case II | 3.88 \cdot 10^{15} | 7.12 \cdot 10^{15} | 5.22 \cdot 10^{15} | 7.13 \cdot 10^{15} |
| Case III | 6.33 \cdot 10^{15} | 9.31 \cdot 10^{15} | 1.12 \cdot 10^{16} | 1.16 \cdot 10^{16} |

Now we find by the method of Baker and Davenport [1] criteria for which there are no solutions.

*Lemma 9* Suppose \(1000 \leq a \leq a_0\) and put

\[ \delta_1 := \frac{\log |\theta_{i,j,k}^{(\sigma - i)/\mu}|}{\log |\alpha_2|} \quad \text{and} \quad \delta_2 := \frac{\log |\alpha_1|}{\log |\alpha_2|}, \]

where \(i\) and \(k\) are chosen according to (20), (21) and (22). Further let \(\delta_1\) and
Let $\delta_2$ be rationals such that
\[ |\delta_1 - \tilde{\delta}_1| < 10^{-60} \quad \text{and} \quad |\delta_2 - \tilde{\delta}_2| < 10^{-60} \]
and assume there exists a convergent $p/q$ in the continued fraction expansion of $\delta_2$, with $q \leq 10^{30}$ and
\[ q \|q \tilde{\delta}_1\| > 1.0001 + \frac{c_6}{a \log a}, \]
then there is no solution for the case corresponding to $j$, $\mu$ and $B_1 \neq 0$. The quantities $c_6$ are listed in table 10.

**Table 10**

<table>
<thead>
<tr>
<th>$c_6 = \mu$</th>
<th>$\mu \sim 1$</th>
<th>$\mu \sim \alpha_1 - 1$</th>
<th>$\mu \sim \alpha_2 - 1$</th>
<th>$\mu \sim \alpha_3 - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>1.9831 · 10^{16}</td>
<td>2.329 · 10^{16}</td>
<td>1.9818 · 10^{16}</td>
<td>1.7907 · 10^{16}</td>
</tr>
<tr>
<td>Case II</td>
<td>7.7806 · 10^{15}</td>
<td>1.425 · 10^{16}</td>
<td>9.5035 · 10^{15}</td>
<td>1.5862 · 10^{16}</td>
</tr>
<tr>
<td>Case III</td>
<td>1.4082 · 10^{16}</td>
<td>2.2395 · 10^{16}</td>
<td>2.2459 · 10^{16}</td>
<td>2.5916 · 10^{16}</td>
</tr>
</tbody>
</table>

**Proof:** We give the details for our standard case I ($\mu \sim \alpha_2 - 1$). The other cases are similar.

Assume that there is a solution corresponding to case I ($\mu \sim \alpha_2 - 1$) such that $B_1 \neq 1$. From (19) we have
\[ |\delta_1 + B_1 \delta_2 + B_2| \leq \frac{2c_4}{|Y_0|^3 \log |\alpha_2|} \leq \frac{8.075a^2}{\exp(21.4827a)} < 10^{-1000}. \]

Multiplication by $q$ yields
\[ |q \tilde{\delta}_1 + q(\delta_1 - \tilde{\delta}_1) + B_1(\tilde{\delta}_2q - p) + B_1q(\delta_2 - \tilde{\delta}_2) + B_1p + B_2q| < 10^{-970}, \]
and therefore
\[ ||q \tilde{\delta}_1|| < 10^{-970} + q10^{-60} + |B_1||\tilde{\delta}_2q - p| + |B_1|q10^{-60}. \]

By another multiplication with $q$ we get
\[ q ||q \tilde{\delta}_1|| < 10^{-940} + q^210^{-60} + |B_1||q\tilde{\delta}_2q - p| + |B_1|q^210^{-60} < 1 + 10^{-940} + 2|B_1|. \]

Table 4 and table 9 together with some estimations yield
\[ q ||q \tilde{\delta}_1|| < 1.0001 + \frac{2|B_2|}{0.8989002219a \log a} < 1.0001 + \frac{1.9818 \cdot 10^{16}}{a \log a}. \]

\[ \blacksquare \]
Using Lemma 9 we find:

**Proposition 3** There are no primitive solutions \((X, Y)\) to \((5)\) with \(Y > 1\), provided \(a \geq 1000\).

**Proof:** In each case and each \(\mu\) from table 3 we check by computer for each value of \(a\) in question whether the criteria given in Lemma 9 is fulfilled or not. Combining the result of this computer search with Proposition 2 we obtain the statement of the proposition. For more details on the implementation see Section 8.

\[\blacksquare\]

By part \((5)\) of Lemma 2 and Proposition 3 it is left to solve the Thue equations

\[
X^3(a^2 + a + 7) - XY^2(a^2 + a + 7) - Y^3(2a + 1) = \pm 1,
\]

\[
X^3(a^2 + a + 7) - XY^2(a^2 + a + 7) - Y^3(2a + 1) = \pm (2a + 1),
\]

for \(0 \leq a \leq 999\). Solving these 3996 Thue equations with PARI yields no further solution. Therefore we have proved our main Theorem 1.

## 8 Computer Search

The computations needed to prove Proposition 2 via Lemma 7 and to prove Proposition 3 via Lemma 9 were implemented in MAGMA. The running times on an Intel Xeon PIII 700MHz processor are collected in table 11.

Finally, we have solved the corresponding equations in the case \(0 \leq a \leq 999\) both in MAGMA and in PARI. For references concerning the computer algebra packages used in this work see [4], [16] and [13].

Table 11

Running times in seconds.

<table>
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<tr>
<th>Case</th>
<th>(B_1 = 0)</th>
<th>(B_1 \neq 0)</th>
<th>(B_1 = 0)</th>
<th>(B_1 \neq 0)</th>
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<td>(\mu \sim 1)</td>
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<td>(\mu \sim \alpha_2 - 1)</td>
<td>(\mu \sim \alpha_3 - 1)</td>
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<td>/</td>
<td>15741</td>
</tr>
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<td>4889</td>
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<td>6766</td>
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References


